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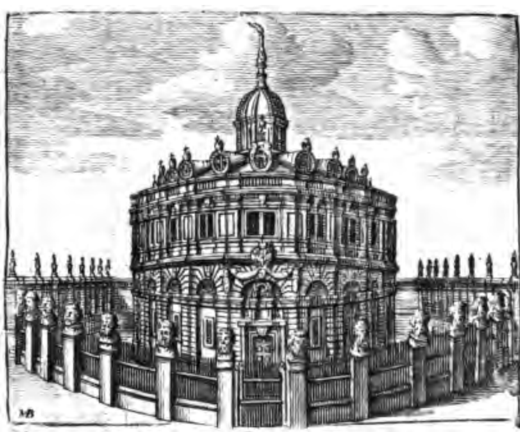
SIVE

Illuſtratio Tractatus D. *Neutoni* De Enumeratione *Linearum Tertii Ordinis*.

Cui Subjungitur,

Solutio Trium Problematum.

Authore JACOBO STIRLING, è Coll. Ball. *Oxon.*



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Imprimatur,

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Apr. 11. 1717.

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HONORATISSIMO Dno.

Dno. *NICOLAO TRON*

EQUITI;

LEGATO Sereniſſimæ Reipublicæ *Venetorum* apud REGEM *Magnæ Britanniæ*.

NON eam præ me fero Inſolentiam, qui primam hanc ſtudiorum meorum ſobolem veſtrum (Vir Illuſtriſſime) patrocinium mereri exiſtimem. Cæterùm hæc mihi prima in manus tradita eſt occaſio agnoſcendi ea Beneficia, quibus immerentem Me accumulare es dignatus. Nihil fortaſsè gratius à Tenuitate noſtrâ Tibi rependi potuit; tum quoniam celeberrimi *Neutoni* Opera maximis Tu, ſi quis alius, in Deliciis ſemper habuiſti, tum quòd Veſtra Excellentia non minus amet,

quam ſoveat Philoſophiam, quæ niſi *Mathematicis* Principiis ſuffulta, corruet. Hic ſeſe mihi aperit Laudum Tuarum Campus, quas aggredi eſſet ſupervacaneum, ſiquidem dulcis Patriæ amor, & inſignis Tua in rebus gerendis Peritia ſatis ſuperque innoteſcunt ex ſublimi Dignitatis gradu ad quem Sereniſſima Reſpublica Te promovit. *Cùm igitur tot & tanta negotia ſuſtineas*, nihil addam ulteriùs, *ne longo ſermone morer Tua tempora*, & nè contra ipſas *Matheſeos* (quam colo) Regulas, brevitatis ſemper amantes, peccare videar.

Excellentiæ Veſtræ

Omni cultu & obſequio

Devinctiſſimus

JA. STIRLING.

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LINEÆ TERTII ORDINIS NEUTONIANÆ, &c.

DEFINITIONES.

1. **L**INEA *Geometrica* eſt cujus Abſciſſæ & Ordinatæ correſpondentes eandem inter ſe ubique obtinent relationem.

2. *Linea Rationalis* eſt cujus Abſciſſæ & Ordinatæ relationem obtinent æquatione vulgari Algebraicâ deſignabilem.

3. *Linea Irrationalis* eſt quando relatio illa æquatione iſtiuſmodi exprimi nequit.

4. *Aſymptotos* curvæ eſt Linea ſimpliciſſima, ſive curva ſive recta ſit, quæ ad curvæ crus tanto magis continue accedit quanto magis producitur, tandem cum eo coincidens.

5. *Curvæ ejuſdem generis* ſunt quæ pro Aſymptotis ſuis ſortiuntur Lineas ejuſdem ſpeciei.

6. *Hyperbola Inſcripta* eſt quæ tota jacet in Aſymptoton angulo ad inſtar Hyperbolæ Conicæ.

7. *Circumſcripta* eſt quæ Aſymptotos ſecat & partes abſciſſas in ſinu ſuo complectitur.

8. *Ambigena* eſt quæ uno crure *Inſcribitur* & altero *Circumſcribitur*.

9. *Conchois* eſt figura habens duo crura ad eaſdem ejuſdem Aſymptoti partes jacentia, & in plagas oppoſitas protenſa, cum vertice verſus Aſymptoton concavo.

10 *Anguinea* vero eſt figura, quando crura jacent ad diverſas Aſymptoti partes.

11. Figura *Cruciformis* eſt, quando quatuor ejus crura in uno puncto conveniunt.

12 *Nodata* eſt, quando duo crura ſe invicem decuſſant: *Nodum* quaſi efficientia.

13. *Cuſpidata* eſt, quando crura in eorum conjunctione *Cuſpidem* efficiunt.

14. *Punctata* eſt quæ *Ovalem* habet conjugatam infinite parvam, id eſt, punctum.

15. *Pura* eſt quæ *Ovali*, *Nodo*, *Cuſpide* & *Puncto conjugato* privatur.

Linearum Rationalium Ordines.

Linearum Rationalium obvia eſt diviſio, ab ipſarum naturis deſumpta, in ſimpliciores ſcilicet & magis compoſitas pro ratione dimenſionum æquationis, quâ relatio inter Abſciſſas & Ordinatas definitur; quando quidem æquatio illa ſimpliciſſima eſt in qua quantitates indeterminatæ ſunt pauciorum dimenſionum. Qua ratione generaliſſima æquatio alicujus ordinis comprehendit omnes lineas ejuſdem. Ergo Linea primi ordinis erit recta ſola æquatione

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$y + ax + b = 0$ deſignata, Eas ſecundi ordinis deſignat $y^2 + \overline{ax+b}\,y + cx^2 + dx + e = 0$, Eas tertii $y^3 + \overline{ax+b}\,y^2 + \overline{cx^2+dx+e}\,y + fx^3 + gx^2 hx + k = 0$, Eas quarti $y^4 + \overline{ax+b} \times y^3 + \overline{cx^2+dx+e} \times y^2 + \overline{fx^3+gx^2+hx+k} \times y + lx^4 + mx^3 + nx^2 + px + q = 0$, Eas quinti $y^5 + \overline{ax+b} \times y^4 + \overline{cx^2+dx+e} \times y^3 + \overline{fx^3+gx^2+hx+k} \times y^2 + \overline{lx^4+mx^3+nx^2+px+q} \times y + rx^5 + sx^4 + tx^3 + ux^2 + vx + w = 0$, Et ſic proceditur in infinitum.

In hiſce æquationibus x eſt Abſciſſa, y Ordinata in quovis angulo ad ſe invicem inclinatæ; $a, b, c, d,$ &c. quantitates datæ ſignis ſuis + & — affectæ, quarum una vel plures deeſſe poteſt, modo ex tali defectu, Linea non migret in aliam ordinis inferioris.

Hæ æquationes ſunt ſui ordinis generaliſſimæ; continent quippe omnes Abſciſſæ & Ordinatæ combinationes, ubi earum dimenſiones in uno æquationis termino ſimul ſumptæ non ſuperant dimenſionem maximam ordinatæ. Etenim dimenſio Curvæ pendet ex maxima dimenſione Abſciſſæ & Ordinatæ in eodem æquationis termino.

Numerus Coefficientium in illis æquationibus.

Per Coefficientes hic intellige quantitates datas $a, b, c, d,$ &c. Harum numerus in prima æquatione eſt 2, in ſecunda 5, in tertia 9, in quarta 14, in quinta 20, & ſic porro. Atque

hi numeri ſic generantur, eſt $5 = 2 + 3$, $9 = 2 + 3 + 4$, $14 = 2 + 3 + 4 + 5$, $20 = 2 + 3 + 4 + 5 + 6$, &c. Adeoque ex Arithmetica ſummatoria univerſaliter facile colligitur, quod ſi ſit n numerus dimenſionum curvæ, erit $\frac{n^2 + 3n}{2}$ numerus Coefficientium in æquatione generaliſſima Lineas omnes illius ordinis definiente. Hujus uſus in ſequentibus patebit.

PROPOSITIO I. THEOREMA.

Omnis Linea Geometrica curvaturâ continuâ vel in ſe redit, vel pergit in infinitum.

Lineam Geometricam motu puncti continuo deſcriptam hic conſidero; omnis enim Linea Geometrica motu puncti certâ quadam conditione conſtanti moventis deſcribi poteſt: quum igitur motus puncti lege immutabili attemperatur, neceſſario durabit ejus motus in infinitum. Unde via percurſa, id eſt, Linea Geometrica, vel in ſe redit, vel pergit in infinitum, idque curvaturâ continuâ ob regularem puncti motum. Q. E. D.

Coroll. 1. Superficies ſolidorum omnium Geometricorum, curvaturâ continuâ, vel in ſe redeunt, vel pergunt in infinitum. Nam ſuperficies Geometricæ, eodem plane modo Linearum motu genitæ concipiendæ ſunt, quo Lineæ punctorum motu: adeoque hoc Corollarium & hæc Propoſitio ſimili prorſus argumentandi genere demonſtrantur.

Coroll.

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Coroll. 2. Crura infinita alicujus Lineæ ductu continuo ſemper conjunguntur. Nam punctum deſcribens neceſſario tranſit ab uno crure ad aliud.

Coroll. 3. Et inde neceſſario ſequitur, quod crurum infinitorum numerus ſemper eſt par: alias enim ſervari nequit motus puncti continuus in infinitum.

Coroll. 4. Omnes rectæ parallelæ ſecant curvam aliquam in iiſdem numero punctis realibus & imaginariis. Hoc Corollarium facillime patet ex Propoſitione & Corollariis ejus ſecundo & tertio.

Coroll. 5. Unde ſi æquatio quælibet involvat duas indeterminatas, Abſciſſam *x* & Ordinatam *y*: numerus valorum poſſibilium & impoſſibilium Ordinatæ *y*, in omni Abſciſſæ *x* magnitudine idem ſemper eſt. Hujus Corollarii beneficio invenire licet numerum radicum æquationis Fluxiones involventis, ut & æquationis in quâ quantitates indeterminatæ indeterminatas habent etiam Exponentes; ut poſtea patebit.

Coroll. 6. Ex Lineæ Geometricæ curvaturâ continuâ, ſequitur nota illa æquationum proprietas; ſcilicet quod radicum impoſſibilium numerus ſemper eſt par.

Serierum infinitarum frequens in ſequentibus erit uſus: viſum eſt igitur aliquid de iiſdem præfari, quum nec earum natura, nec methodus eas inveſtigandi ab aliquo, quod ſciam, huc uſque ſatis explicata fuerit.

De

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De Serierum infinitarum Ortu.

D. *Walliſius*, in *Arithmeticâ Infinitorum* Anno 1655. publicatâ, multis exemplis particularibus generaliter tandem invenit, quod ſi Ordinata curvæ ſit $x^{\frac{m}{n}}$ erit ejus area $\frac{n}{m+n}$ $x^{\frac{m+n}{n}}$ Ope hujus regulæ quadravit omnes curvas quarum Ordinatas habere potuit in terminis rationalibus expreſſas. D. *Neutonus* per Interpolationem arearum ab Ordinatis (per *Walliſii* regulam) deductarum quadravit Circulum. Et ex datâ ejus areâ in ſerie infinitâ, per reverſum regulæ *Walliſii* invenit ejus ordinatam in ſerie etiam infinita expreſſam. Et methodum Interpolandi proſecutus, Theorema ſuum invenit pro elevando Binomio ad dignitatem quamvis indeterminatam: ut conſtat ex Epiſtolâ ejus ad D. *Oldenburgium* 13 Junii Anno 1676. miſſa. Sed Interpolationum methodum miſſam tandem faciens, operationes ſpecioſas perinde ut Arithmeticas inſtituere cœpit, atque docuit reducere radices æquationum omnium, primo ſimplicium deinde affectarum, in ſeries convergentes. Hoc patet ex ejus *Analyſi* à *Barrovio* ad *Collinium* menſe Julio, Anno 1669. miſſa. In eâdem *Analyſi*, Serierum ope, quadravit curvas tum *Geometricas* tum *Mechanicas* ut appellari ſolent. Et docuit quâ ratione, ex datâ areâ vel longitudine curvæ inveniri poſſit

Ba-

Baſis vel Ordinata. Sub finem ejuſdem *Analyſeos*, ope ſerierum univerſaliter demonſtravit regulam *Walliſii*, methodo nova quæ alia non erat, quam *Fluxionum* methodus.

Carteſius, *Barrovius*, aliique in *Tangentium* methodis, docuerunt invenire rationes primas & ultimas quantitatum *Naſcentium* & *Evaneſcentium*, at non generaliter; Et *Walliſius*, uti mox dictum eſt, oſtendit quomodo inveniri poſſit area ex data Ordinata terminis rationalibus expreſſa: Hæc erant dubia & obſcura veſtigia *Fluxionum* methodi directæ & inverſæ. Et impoſſibile fere erat, abſque ſerierum doctrina, hanc methodum ulterius promovere; quam promoverunt præfati docti Viri. Unde ſane non video quâ ratione quis poſſit *Fluxionum* methodi inventionem ſibi arrogare, & non etiam ſerierum inventionem.

De Natura Serierum.

SErierum methodus in eo fundatur, ut primo aſſumatur quantitas radici quæſitæ æqualis quam proxime, & corrigatur valor aſſumptus continue: quo pacto habebitur tandem quantitas, quæ à radicis vero valore minus diſtabit quavis quantitate data. Hoc vero multifariam præſtari poteſt.

Sit Series eo citius convergens quo minor eſt x, ſcilicet $y = Ax^{n} + Bx^{n+r} + Cx^{n+2r} + Dx^{n+3r} +$ &c. Ponamus eſſe x admodum parvam, & terminus quilibet poſterior erit priore multo minor, atque

que termini pauci initiales ad verum ipſius y valorem quam proxime accedent. Quod ſi ſit x infinite parva, erit accurate $y = Ax^n$, terminis reliquis hujus termini reſpectu evaneſcentibus. In æquatione relationem inter x & y definiente, ſuppone x etiam infinite parvam, & termini quidam æquationis evadent reliquis infinite minores, qui proinde reliquorum maximorum reſpectu evaneſcent; è terminis igitur maximis ejuſdem ordinis tanquam nihilo æqualibus, (eodem plane modo quo ex æquatione numerali) extrahe radicem, nam erit illa Ax^n: qui terminus primus propterea dabitur. Per terminos maximos *ejuſdem ordinis* intellige eos qui ad ſe invicem datam habent rationem, & ſunt reliquis omnibus infinite majores. Pone $p = Bx^{n+r} + Cx^{n+2r} + Dx^{n+3r}$ + &c. terminis nondum inventis, & erit $y = p + Ax^n$, quem ipſius y valorem in æquatione ſubſtituendo, obtinebis æquationem novam indeterminatas duas x, p involventem. In illa æquatione nova ſuppone etiam x infinite parvam, ut ſit $p = Bx^{n+r}$ accurate, atque ex terminis maximis ejuſdem ordinis tanquam nihilo æqualibus radix extracta erit Bx^{n+r}, qui terminus ſecundus proinde datur. Sit $q = Cx^{n+2r} + Dx^{n+3r}$ +, &c. Unde erit $p = Bx^{n+r} + q$: hunc valorem ipſius p ſubſtitute in æquatione relationem inter x, p exprimente; & habebis æquationem tertiam quæ oſtendit quam inter ſe x, q obtinent relationem. Ex hac æquatione invenies tertium terminum Cx^{n+2r} eodem plane modo quo terminos duos primos ex æquationibus

duabus prioribus obtinuiſti. Et opus continuando invenire licet terminos ſeriei ſequentes tot quot volueris.

Et ſi eſſet ſeries hujuſmodi $y = Ax^n + Bx^{n-r} + Cx^{n-2r} + Dx^{n-3r} +$ &c. eo citius convergens quo major eſt x; inter operandum ſupponenda eſt x infinite magna: atque terminis ordinum inferiorum rejectis è maximis ejuſdem ordinis educenda eſt radix; nam illa eſt Ax^n. Ponendo $p = Bx^{n-r} + Cx^{n-2r} + Dx^{n-3r} +$ &c. adeoque $y = p + Ax^n$, habebitur æquatio nova, ex qua invenire licet terminum ſeriei ſecundum, & opus continuando reliquos tot quot eſt animus.

Et eadem ratione qua inveniuntur ſeries eo citius convergentes quo major, vel quo minor eſt x, invenire licet ſeries eo citius convergentes quo propius accedit x ad datam quamvis quantitatem. v. g. Si quæratur ſeries eo citius convergens quo propius accedit x ad quantitatem a; nec ſuppono x infinite magnam nec infinite parvam, ſed æqualem ipſi a, & tum quæro valorem ipſius y, nam ille valor eſt primus terminus ſeriei. Et quomodo ad libitum procedendum eſt, ex hactenus traditis ſatis patet.

Hæc de natura ſerierum & fundamento methodi ad eas perveniendi dicta ſufficiant. Proceſſus certe legitimus cuivis in diverſis infinitorum ordinibus quam minime etiam verſato neceſſario patet. Ex ipſa operatione facile videre eſt, has ſeries non dare radices æquationum quæſitas, ni ſat celeriter convergant; etenim

nim totus operandi proceſſus in eo fundatur, ut ſit x ſatis parva vel ſatis magna, hoc eſt, ut termini ſeriei ſubſequentes ſint antecedentibus perpetuo minores. Hinc hallucinantur ii qui ſe aliquid Geometrice etiam accuratum ex ſeriebus parum convergentibus vel aliquando quidem divergentibus collegiſſe ſomniant. Eodemque modo demonſtratur Diviſio & Radicum Extractio Arithmetica. Omnes enim hujuſmodi operationes tam Arithmeticæ quam Specioſæ eodem innituntur fundamento; ſcilicet ut ſeriei termini initiales ad quæſitum quam proxime accedant, reliqui vero ut eo magis continue ſint minores quo magis à primo diſtant. Qui ſeries divergentes adhibent in Problematum ſolutione; idem faciunt, ac ſi Diviſionem Arithmeticam verſus dextram, non verſus ſiniſtram inchoarent. Iſtiuſmodi enim ſeries quæſitum nunquam dant, ſed mere ſunt imaginariæ.

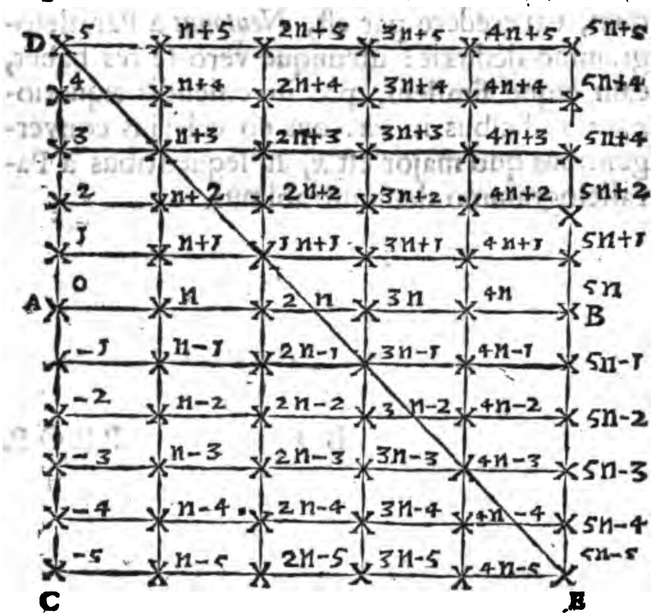
Hiſce jam expoſitis, ſerierum inventio couſque reducitur, ut inveniantur termini maximi alicujus æquationis ejuſdem ordinis; poſito quod una indeterminatarum quas involvit æquatio, evadit infinite magna vel infinite parva. Quod tamen perficere nemo unquam valuit præter D. *Neutonum* ſerierum Inventorem. Duplici utitur ille methodo, una, Parallelogrammi ſcilicet, quam deſcripſit in Epiſtola ad D. *Oldenburgium* 24 Octob. Anno 1676. miſſa; quæ, quamvis à plurimis qui eam haud intellexerunt, ut methodus Mechanica diu neglecta jacuit, eſt om-

omnium quam quis excogitare poteſt generaliſſima & elegantiſſima. Alteram methodum deſcripſit D. *Neutonus* in Epiſtola ad D. *Wallisium* Anno 1692 miſſa: quæ, quamvis Geometræ præ priore hucuſque amplexi ſunt, eſt particularis ſolummodo; utpote cujus ope tantum invenire licet ſeries eo citius convergentes quo minor eſt *x*; & hoc non adeo generaliter. Hæc methodus ait *Neutonus*, *ejuſdem eſt generis cum illa pro extrahendis radicibus ex æquationibus affectis ſuperius deſcripta.* Hanc igitur methodum, uti credere par eſt, *Neutonus* à Parallelogrammo deduxit: utcunque vero ſe res habet, eam eique ſimilem, pro inveniendis æquationum radicibus in ſeriebus eo celerius convergentibus quo major eſt *x*, in ſequentibus à Parallelogrammo demonſtrabimus.

PROP. II. PROBL.

Si una variabilium quas involvit æquatio evadat infinite magna vel parva, opporteat invenire terminos illius æquationis maximos ejuſdem Ordinis.

DUC rectam DAC eique ad rectos angulos AB: haſce rectas divide in lineolas innumeras æquales, à quarum rectularum extremitatibus erige normales diſtribuentes ſpatia angularia DAB, CAB in rectangula innumera æqualia.



Sit x indeterminata ex cujus poteſtatibus conficienda eſt ſeries, n index ipſius x in primo termino ſeriei, adeo ut ſit radix quæſita y æqualis

lis Ax^r accurate, cum x eſt infinite magna vel infinite parva, prout ſeriei quæſitæ natura requirit.

Concipe rectæ CD partes ſingulas datas eſſe & unitati æquales, rectæ vero AB partes ſingulas variabiles & quantitati n ſemper æquales. In punctis angularibus æqualium rectangulorum ſubſtitue dignitates quantitatum x, x^n regulariter aſcendentium & deſcendentium à puncto A, uti in Schemate videre eſt.

Index cujuſvis poteſtatis ipſius x in recta CD poſitæ æqualis eſt ejus diſtantiæ à puncto A: affirmativi ſunt indices ſupra punctum A, negativi vero qui infra locantur. Eodemque modo, quum pars quælibet rectæ AB æqualis ſit n, index dignitatis cujuſvis ipſius x in recta AB poſitæ dat ejus diſtantiam à puncto A. Adeoque in punctis angularibus extra rectas AB & CD poſitis, una pars indicis dat diſtantiam à recta AB, altera pars diſtantiam à recta CD. Et inde index totus ſimul ſumptus æqualis eſt ſummæ aut differentiæ talium diſtatiarum, prout jacet ſupra vel infra rectam AB.

Duc jam rectam quamvis DE tranſeuntem per puncta duo quævis angularia x^r, x^{s+n-3}. Et indices terminorum omnium quos attingit recta illa DE erunt æquidiſtantes in progreſſione Arithmetica: Supponamus hoſce indices ſibi invicem æquari, hoc eſt, eorum differentiam eſſe nihil, & indices omnes alii majores, minores vel æquales erunt horum indicum alterutri, prout jacent ſupra, infra vel in ipſa recta

DE.

DE. Hæc patent ex ſummis vel differentiis rectarum quibus diximus omnes indices æquari reſpective. Igitur ſi x ſit quantitas infinite magna, termini quos attingit recta DE, erunt ejuſdem ordinis, reliqui vero infinite majores vel minores terminis attactis prout jacent ſupra vel infra rectam DE. Et ſi x ſit quantitas infinite parva, termini quos attingit DE erunt ejuſdem ordinis, reliqui vero erunt terminis attactis infinite majores vel minores prout jaceant infra vel ſupra rectam DE. Atque hinc tandem oritur ſolutio Problematis: ſcilicet quum æquatio aliqua proponitur, pone $y = x^n$, $y = x^{n-1}$, $y = x^{n-2}$, $y = x^{n-3}$, &c. neglectis coefficientibus. Terminos æquationis reſultantes eodem modo diſpone, quo in Schemate, cuique locum proprium adſcribendo pro ratione indicis: ad horum terminorum ſic diſpoſitorum duos vel plures applicetur regula, ita ut omnes reliqui cadant ſupra vel infra regulam, prout quæris ſeriem ex aſcendentibus vel deſcendentibus ipſius x dignitatibus confectam. Et termini quos attingit regula erunt maximi ejuſdem ordinis. Q. E. I.

Si quando forte accidit, quod indices ipſius x ſint fracti, vel etiam ſi vis ſurdi, & nimis operoſum foret eos tollere; ſubdividendæ ſunt partes lineæ CD: & inde erigendo normales, in earum cum reliquis concurſu diſponendæ ſunt poteſtates quarum indices fracti ſunt vel ſurdi. Hujus eadem ac Propoſitionis eſt demonſtratio.

Coroll.

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Coroll. 1. Supponamus terminos omnes infra rectam DE abeſſe. Et (per hanc Propoſitionem) termini x^5, x^{n+3}, x^{2n+1}, x^{3n-1}, x^{4n-3} & x^{5n-5} quos attingit Regula, erunt maximi ejuſdem ordinis modo ſupponatur x infinite parva; & inde eorum indices neceſſario æquantur; hoc eſt, $n+3=5$, $2n+1=5$, $3n-1=5$, $4n-3=5$, & $5n-5=5$: harum quinque æquationum quælibet dabit $n=2$. Æquetur jam numerus 5 indici termini cujuſvis alius ſupra rectam DE poſiti; ſit exempli gratiâ $2n+3=5$, erit $n=1$; ſit $4n=5$, erit $n=\frac{5}{4}$; ſit $5n+5=5$, erit $n=0$; ſit $3n+4=5$, erit $n=\frac{1}{3}$. Vides igitur quod æquando omnes indices numero 5 (qui hic ponitur index infimæ dignitatis ipſius x) valor ipſius n verus, eſt omnium valorum ſic prodeuntium maximus, reliqui tamen vero valore ſemper prodeunt minores.

Coroll. 2. Cum itaque æquatio aliqua proponitur, & quæritur ejus radix in ſerie eo citius convergente quo minor eſt x, ſit λ index infimæ poteſtatis ipſius x quæ nec per y, $\dot{y}$, $\ddot{y}$, $\dddot{y}$, &c. aut earum dignitates aliquas multiplicatur; hoc eſt, ſit λ index infimæ dignitatis ipſius x in rectâ CD poſitæ; pone $y=x^n$, $\dot{y}=x^{n-1}$, $\ddot{y}=x^{n-2}$, &c. Et, hiſce valoribus in æquatione ſubſtitutis, æquentur omnes indices terminorum ſic reſultantium indici λ, & valor ipſius n omnium maximus inde proveniens, erit index ipſius x in primo termino ſeriei quæſitæ.

Eſt-

Eſtque hæc altera D. *Neutoni* methodus pro inveniendo indice termini primi.

Coroll. 3. Supponamus jam terminos omnes infra rectam DE jacere, & reliquos abeſſe, ς qui nunc eſt index altiſſimæ dignitatis earum quæ nec per $\dot{y}$, $\ddot{y}$, $\dot{y}$, $\ddot{y}$, &c. aut earum poteſtates aliquas multiplicantur, æquatus indici cujuſvis termini in rectâ DE poſiti ſemper dabit $n=2$, ut antea in *Corollario* primo. Æquetur autem ς indici dignitatis alicujus ipſius x infra rectam DE jacentis; ſit verbi gratia, $n+1=\varsigma$, erit $n=4$; Sit $2n-3=\varsigma$, erit $n=4$; ſit $n-3=\varsigma$ & erit $n=8$. Conſtat ergo quod 2 verus ipſius n valor, eſt omnium ſic provenientium ſemper minimus. Unde duco regulam ſequentem.

Coroll. 4. Si quæritur ſeries eo citius convergens quo major eſt x, ſit $y=x^n$, $\dot{y}=x^{n-1}$, $\ddot{y}=x^{n-2}$, &c. hoſce valores in æquatione ſubſtitue, & terminorum omnium reſultantium indices æquentur ipſi λ (indici altiſſimæ dignitatis ipſius x, quæ nec per $\dot{y}$, $\ddot{y}$, $\dot{y}$, $\ddot{y}$, &c. aut earum dignitates aliquas multiplicatur) atque valor ipſius n omnium minimus hoc modo inventus, eſt index ipſius x in primo ſeriei termino.

Methodus hæc pro inveniendo indice primi termini ſeriei ex deſcendentibus ipſius x dignitatibus confectæ, ſimilis eſt methodo D. *Neutoni* in Corollario ſecundo expoſitæ, pro inveniendo

cado indice termini primi in ſerie ubi poteſtates ipſius x perpetuo ſunt aſcendentes; earum vero neutra haud adeo generalis eſt, uti ex Parallelogrammo facile cuivis videre eſt.

Coroll. 5. Terminos quos attingit recta DE appello *primi ordinis terminos*. Moveatur recta illa DE motu parallelo, & ſimul tanget terminos x^4, x^{n+2}, x^{2n}, x^{3n-2}, x^{4n-4}: hi ſunt *termini ſecundi ordinis*; x^3, x^{n+1}, x^{2n-1}, x^{3n-3}, x^{4n-5} ſunt *tertii ordinis termini*; x^2, x^n, x^{2n-2}, x^{3n-4} ſunt *quarti ordinis*: & ſic in infinitum. Terminos enim *ejuſdem ordinis* recta DE motu parallelo lata ſimul tanget. Et ſicut radix extracta ex terminis primi ordinis, dat terminum ſeriei primum; ſic radix extracta ex terminis primi & ſecundi ordinis, dat terminos duos primos, radix extracta ex terminis primi, ſecundi & tertii ordinis dat terminos tres primos; & ſic in infinitum. Unde ſi in æquatione deſint termini ordinum aliquorum intermediorum, termini ſeriei reſpectivi habebuntur extrahendo radicem ex terminis æquationis ordinum ſuperiorum. Deſint, verbi gratiâ, termini tertii, quarti, quinti & ſexti ordinis, & radix extracta ex terminis primi & ſecundi ordinis dabit primos ſex ſeriei terminos. Hæc obſervatio operationi aliquando compendium ſubminiſtret: exemplis vero in ſequentibus illuſtrabitur.

Coroll. 6. Ex hâc Propoſitione invenire licet numerum radicum, quas æquatio Fluxiones involvens habere poteſt, & quibus plures habere

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nequit. Nam termini maximi ejuſdem ordinis dant valorem ipſius y cum x eſt infinite magna vel infinite parva. Et (per Coroll. 5. Prop. 1.) numerus valorum ipſius y in omni Abſciſſæ x magnitudine idem ſemper eſt. Ergo æquationis, quæ dat y, cum x eſt infinite parva vel infinite magna, numerus radicum, æqualis eſt numero radicum quas æquatio propoſita habere poteſt, & quibus plures habere nequit.

Adeoque etiam innoteſcit numerus radicum æquationis hujuſmodi $y^x + ax^z = a$, ubi indeterminatarum indeterminatæ ſunt coefficientes; nam ejuſmodi æquatio tranſmutari poteſt in Fluxionalem.

Aliquando accidit, quod ad inveniendum primum terminum ſeriei, prodeunt duæ æquationes diverſarum dimenſionum: in illis caſibus dimenſio maxima ſemper dat numerum radicum æquationis propoſitæ.

Exemplum Primum.

ÆQuationis $y^3 - a^2y + axy - x^3 = 0$ extrahendæ ſint radices. Ut inveniatur index termini primi ſeriei quæſitæ, pone $y = x^n$, eritque $y^3 = x^{3n}$, $xy = x^{n+1}$. Terminis hiſce dicto modo diſpoſitis in punctis angularibus Parallelogrammi, video exteriorum tres caſus accidere: ſcilicet, ſunt x^3, x^n; x^n, x^{3n}, vel x^3, x^{3n} termini reliquis exteriores. Æquentur indices, & erit 1mo. $n=3$, 2do. $n=0$, & 3^{tio}. $n=1$; igitur

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tur ſi quæritur ſeries ex dignitatibus aſcendentibus confecta, poteſt eſſe 3 vel o index ipſius x in primo termino ſeriei. Moveatur recta $x^n x^3$ motu parallelo, & ea primo perveniet ad x^{n+1} terminum unicum ſecundi ordinis, ſecundo perveniet ad angulum a, tertio ad b, c ſimul, quarto ad d, quinto ad e & ultimo ad x^{3n} terminum ordinis infimi. Cum igitur deſint termini ordinis tertii, quarti, quinti & ſexti: invenientur primi ſex ſeriei termini (per *Coroll.* 5. Prop. 2.) extrahendo radicem ex terminis $-a^2y + axy - x^3$ primi & ſecundi ordinis poſitis nihilo æqualibus. Hoc vero per Diviſionem fit, nam ſi ſit æquatio $-a^2y + axy - x^3 = 0$, erit $y = \frac{x^3}{-a^2 + ax}$ adeoque dividendo eſt $y = \frac{-x^3}{a^2} - \frac{x^4}{a^3} - \frac{x^5}{a^4} - \frac{x^6}{a^5} - \frac{x^7}{a^6} - \frac{x^8}{a^7}$ &c. hi ſunt primi ſex termini per diviſionem inventi: terminus ſeptimus invenietur eſſe $\frac{-2x^9}{a^8}$. Hic diviſionem inchoavi à termino a^2 quoniam ponitur x admodum parva.



In caſu ſecundo erat $n = o$, pro y itaque in æquatione ſubſtitue Ax^o, vel quod idem eſt A, ubi A eſt quantitas determinanda ſtatim invenienda. Et orietur $A^3 - a^2A + axA - x^3 = o$, evaneſcat jam x & erit $A^3 - a^2A = o$, unde eſt $A = o$, vel $A = \pm a$: igitur ter-

minus primus poteſt eſſe $+a$ vel $-a$: valor p ſpectat ad radicem jam inventam. Pone p æqualem terminis nondum inventis & erit $y = a + p$ quem ipſius y valorem in æquatione ſubſtituendo orietur æquatio $p^3 + 3ap^2 + \overline{2a^2 + ax} \times p + a^2x - x^3 = 0$ indeterminatas x, p involvens. Ad inveniendum terminum primum valoris radicis p, ope rectanguli habebimus æquationes $p^3 + 3ap^2 + 2a^2p = 0$, $2a^2p + a^2x = 0$, æquationis illius radices 0, $-a$, $-2a$, ad ſeriem quæſitam non pertinent (ut poſtea explicabitur) hujus vero radix $-\frac{1}{2}x$ eſt terminus ſeriei ſecundus. Pone igitur $p = q - \frac{1}{2}x$, & orietur $q^3 + \overline{3a - \frac{3}{2}x} \times q^2 + \overline{2a^2 - 2ax + \frac{3}{4}x^2} \times q + \frac{1}{4}ax^2 - \frac{9}{8}x^3 = 0$, ubi æquatio $2a^2q + \frac{1}{4}ax^2 = 0$ dat $q = -\frac{x^3}{8a}$ proxime, vel $q = r - \frac{x^2}{8a}$. Hunc valorem pro q ſubſtituo, neglectis terminis q^3, $-\frac{3}{2}xq^2$, ut & iis in quibus r erit pluſquam unius dimenſionis, quos nulli uſui futuros ſatis indicabit ipſa operatio, modo unum vel duos forte ſeriei terminos plures quærere tantum eſt animus; & exurget $2a^2r - 2axr = \frac{7}{8}x^3 - \frac{3x^4}{64a}$ proxime, atque dividendo, erit $r = \frac{7x^3}{16a^2} + \frac{59x^4}{128a^3}$ &c. Ergo $y = a + p = a - \frac{1}{2}x + q = a - \frac{1}{2}x - \frac{x^3}{8a} + r = a - \frac{1}{2}x - \frac{x^2}{8a} + \frac{7x^3}{16a^2} + \frac{59x^4}{128a^3}$ &c. Et ſi pro primo

termino uſurpaſſem $-a$, prodiiſſet $y = -a + \frac{1}{2}x + \frac{x^2}{8a} + \frac{9x^3}{16a^2} + \frac{69x^4}{128a^3}$ &c.

Hic notandum eſt quod primi tres termini prodeunt rejiciendo x^3, & ex terminis reliquis $y^3 - a^2y + axy = 0$, id eſt, ex $y^2 - a^2 + ax = 0$ extrahendo radicem.

Radix unica in ſerie eo citius convergente quo major eſt x.

In tertio caſu erat n unitas; pone ergo $y = Ax$, & orietur $A^3x^3 - a^2Ax + aAx^2 - x^3 = 0$, termini maximi $A^3x^3 - x^3$ ejuſdem ordinis poſiti æquales nihilo dant $A = 1$; ergo eſt x primus terminus ſeriei. Pone $y = x + p$, & reſultabit $p^3 + 3xp^2 + \overline{3x^2 + ax - a^2} \times p + ax^2 - a^2x = 0$, ubi termini $3x^2p$, ax^2 poſiti æquales nihilo dant $p = -\frac{1}{3}a$ fere, vel $p = q - \frac{1}{3}a$: hunc ipſius p valorem ſubſtituendo, reſultabit æquatio $q^3 + \overline{3x - a} \times q^2 + \overline{3x^2 - ax - \frac{2}{3}aa} \times q - \frac{8}{27}a^2x + a^3 = 0$, ubi termini $3qx^2$, $-a^2x$ dant $\frac{aa}{3x}$ pro termino tertio. Pone $q = r + \frac{aa}{3x}$, & illum valorem ſubſtituendo, neglectis terminis nulli uſui futuris, habebis $\overline{3x^2 - ax} \times r = \frac{8}{27}a^3 - \frac{a^4}{9x}$ fere, unde $r = \frac{a^3}{81x^2} - \frac{8a^4}{243x^3}$ &c. Adeoque eſt

$$y = x - \frac{1}{3}a + \frac{a^2}{3x} + \frac{a^3}{81x^2} - \frac{8a^4}{243x^3} \text{ \&c.}$$

Exem-

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Exemplum Secundum.

IN hoc Exemplo indices reperiuntur ope *Cor.* 4. *Prop.* 2.

Ex æquatione $x^3\dot{y} + ayxx + a^2xx - 2a^3x = 0$ extrahendo ſit radix y in ſerie ubi indices ipſius x perpetuo magis deſcendunt. Fluat x uniformiter, & ſit $\dot{x} = 1$, atque æquatio evadet $x^3\dot{y} + axy + a^2x - 2a^3 = 0$. Pone $y = Ax^n$, erit $\dot{y} = nAx^{n-1}$, atque proveniet æquatio $nAx^{n-2} + aAx^{n+1} + a^2x - 2a^3 = 0$. Terminus jam altiſſimæ dignitatis in quo nec $\dot{y}$ neque y reperitur eſt a^2x, ubi index ipſius x eſt unitas: æquentur igitur indices terminorum reliquorum unitati, eritque 1mo, $n+2=1$, unde $n=-1$; 2do, erit $n+1=1$, unde $n=0$, quorum valorum minimus -1, eſt index ipſius x in primo termino ſeriei, & termini nAx^{n+2}, a^2x, vel $-Ax$, $+a^2x$ poſiti æquales nihilo dant $A = a^2$, unde eſt $\frac{a^2}{x}$ primus terminus ſeriei.

Operatio Secunda.

PRO terminis reliquis pone p, & erit $y = p + a^2x^{-1}$, $\dot{y} = \dot{p} - a^2x^{-2}$, unde oritur $x^3\dot{p} + axp - a^3 = 0$: ubi p, $\dot{p}$, vices ſubeunt ipſarum y, $\dot{y}$ in æquatione prima. Sit $p = Ax^n$, $\dot{p} = nAx^{n-1}$; & prodibit $nAx^{n+2} + aAx^{n+1} - a^3 = 0$; terminus unicus in quo nec p neque $\dot{p}$ reperitur eſt a^3, ubi index ipſius x eſt 0; ſit igitur $n+2 = 0$, & erit $n = -2$; Sit $n+1 = 0$, & erit $n = -1$; quo

quorum numerorum minimus -2 eſt index ipſius x in ſecundo termino; & termini nAx^{n+2}, $-a^3$ poſiti æquales nihilo dant $A = \frac{1}{n}a^3x^{-2-n} = -\frac{1}{2}a^3$; ergo terminus ſecundus eſt $-\frac{a^3}{2x^2}$.

Operatio Tertia.

Pone $p = q - \frac{1}{2}a^3x^{-2}$, erit $\dot p = \dot q + a^3x^{-3}$, quos valores ſubſtituendo, orietur æquatio $x^3\dot q + axq - \frac{1}{2}a^4x^{-1} = 0$. Pone $q = Ax^n$; $\dot q = nAx^{n-1}$, & exurget $nAx^{n+2} + aAx^{n+1} - \frac{1}{2}a^4x^{-1} = 0$. Terminus unicus in quo nec q neque $\dot q$ reperitur eſt $-\frac{1}{2}a^4x^{-1}$, ubi index ipſius x eſt -1; jam ponendo $n+2 = -1$, erit $n = -3$; & ponendo $n+1 = -1$, erit $n = -2$, quorum minimus -3 eſt index quem quærimus, & termini nAx^{n+2}, $\frac{1}{2}a^4x^{-1}$, (quorum indices inter ſe æquati dant verum ipſius n valorem) poſiti æquales nihilo dant $A = -\frac{1}{6}a^4$ & inde terminus tertius eſt $-\frac{a^4}{6x^3}$.

Operatio Quarta.

SIT $q = r - \frac{1}{6}a^4x^{-3}$, $\dot q = \dot r + \frac{1}{2}a^4x^{-4}$, & evadet æquatio $x^3\dot r + axr - \frac{1}{6}a^5x^{-2} = 0$, ex qua æquatione invenies terminum quartum eſſe $-\frac{a^5}{24x^4}$. Eſt

igi-

igitur $y = \frac{a^2}{x} - \frac{a^3}{2x^2} - \frac{a^4}{6x^3} - \frac{a^5}{24x^4} -$ &c. hoc eſt,

$$y = \frac{a^2}{1.x} - \frac{a^3}{1.2.x^2} - \frac{a^4}{1.2.3.x^3} - \frac{a^5}{1.2.3.4.x^4} - \frac{a^6}{1.2.3.4.5.x^5} \text{ \&c.}$$

Methodus hæc ſigillatim inveniendi terminos eſt admodum generalis, at plerumque nimis operoſa. Eſt autem alia Methodus haſce radices extrahendi, quæ conſiſtit in aſſumptione ſeriei univerſalis hujuſmodi $y = Ax^n + Bx^{n+r} + Cx^{n+2r} + Dx^{n+3r} +$ &c. & inde determinando Exponentes n, r, & Coefficientes $A, B, C, D,$ &c. Hujus Methodi, longo tempore poſtquam D. *Newtono* innotuit, in Actis Eruditorum *Lipſiæ* D. *Leibnitius*, ſuo etiam nomine edidit Exemplum unum aut alterum in caſibus facilioribus, ubi tantum docuit Coefficientium inventionem; at in indicum non in coefficientium inventione jacebat difficultas. Ideoque D. *Taylor*, in *Prop.* 9. Methodi *Incrementorum*, priuſquam Coefficientes determinat, formam ſeriei invenire aggreditur. Eſtque ut ſequitur.

Inveniatur (per *Prop.* 2. vel ejus *Coroll.*) index termini primi, vocetur is n, in æquatione pro $y, \dot{y}, \ddot{y}, \dddot{y}$, &c. ſcribe x^n, x^{n-1}, x^{n-2}, x^{n-3}, &c. reſpective, adeo ut termini reſultantes componantur omnes ex x & datis quantitatibus: ſit r maximus communis diviſor indicum terminorum ſic reſultantium, & forma ſeriei erit hæc $y = Ax^n + Bx^{n+r} + Cx^{n+2r} + Dx^{n+3r} +$ &c. Erit vero r negativa aut affirmativa, prout quæris ſeriem

ex-

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ex dignitatibus ipſius x deſcendentibus aut aſcendentibus confectam.

Exemplum Tertium.

ÆQuationis $y^3 + axy - x^3 = 0$, quæratur radix cum x eſt admodum magna. Invenio (per *Prop.* 2.) unitatem eſſe indicem ipſius x in primo termino ſeriei: pone igitur $y = x$, & æquatio evadet $x^3 + ax^2 - x^3 = 0$. Horum indicum 2, 3, 3 maximus communis diviſor eſt unitas, ergo ſeriei forma erit $y = Ax + B + Cx^{-1} + Dx^{-2} + Ex^{-3} +$ &c. Sequitur operatio.

$$
\begin{array}{r|l}
y^3 = & A^3x^3 + 3A^2Bx^2 \left.\begin{array}{l} + 3A^2C \\ + 3AB^2 \end{array}\right\} x \left.\begin{array}{l} + 3A^2D \\ + 6ABC \\ + BBB \end{array}\right\} x^0 \\
 & \left.\begin{array}{l} + 3A^2E \\ + 6ABD \\ + 3AC^2 \\ + 3B^2C \end{array}\right\} x^{-1} + \&c. \\
 & \\
- x^3 = & - x^3 \\
 & \\
+ axy = & \quad * \quad + aAx^2 \quad + aBx \quad + aCx^0 \\
 & + aDx^{-1} + \&c.
\end{array}
$$

Comparando Coefficientes terminorum homologorum, erit $A^3 = 1$, unde $A = 1$. $3A^2B + aA = 0$, unde $B = -\frac{1}{3}a$, $3A^2C + 3AB^2 + aB = 0$, unde $C = 0$. $3A^2D + B^3 = 0$, unde $D = \frac{1}{81}a^3$. $3A^2E + 6ABD + aD = 0$, unde $E = \frac{1}{243}a^4$. &c. Ergo $y = x - \frac{1}{3}a + \frac{a^3}{81x^2} + \frac{a^4}{243x^2} +$ &c.

 Exem-

Exemplum Quartum.

EX æquatione $x^2y^2 - 3x^2xy + 2x^2x^2 - axy^2 + a^2x^2 = 0$, extrahenda ſit radix y in ſerie eo citius convergente quo major eſt x. Invenio formam ſeriei fore $y = Ax + B + Cx^{-1} + Dx^{-2} + Ex^{-3} +$ &c. Fluat x uniformiter, exiſtente $\dot{x} = 1$, ut in caſibus hujuſmodi vulgo fit; & evadet æquatio $y^2 - 3x^2y + 2x^2 - axy^2 + a^2 = 0$. Sequitur operatio.

$$
\begin{array}{r|l}
y^2 = & A^2x^2 + 2ABx \left.\begin{array}{l} +2AC \\ +2AB \end{array}\right\} x^0 \left.\begin{array}{l} +2AD \\ +2BC \end{array}\right\} x^{-1} \\
& \left.\begin{array}{l} +2AE \\ +2BD \\ +CC \end{array}\right\} x^{-2} + \&c. \\
-3x^2y = & -3Ax^2 \quad * \quad +3Cx^0 \quad +6Dx^{-1} \\
& +9Ex^{-2} + \&c. \\
+2x^2 = & +2x^2 \\
-axy^2 = & * \quad -aA^2x \quad * \quad -2aACx^{-1} \\
& -4aADx^{-2} - \&c. \\
+a^2 = & * \quad * \quad +a^2x^0
\end{array}
$$

Ex comparatione Coefficientium invenio

$A = 1$, $B = \frac{1}{2}a$, $C = -\frac{1}{4}aa$, $D = -\frac{1}{32}a^3$, $E = -\frac{5}{352}a^4$, &c.

$A = 2$, $B = a$, $C = -\frac{2}{7}aa$, $D = -\frac{2}{35}a^3$, $E = -\frac{104}{3185}a^4$, &c.

$$
\text{Ergo } y = \begin{cases} x + \frac{1}{2}a - \dfrac{a^2}{4x} - \dfrac{a^3}{32x^2} - \dfrac{5a^4}{352x^3} \ \&c. \\[2ex] 2x + a - \dfrac{2a^2}{7x} - \dfrac{2a^3}{35x^2} - \dfrac{104a^4}{3185x^3} \ \&c. \end{cases}
$$

Radi-

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Radices ejuſdem æquationis cum x *eſt valde parva.*

Invenietur forma ſeriei hæc, $y = Ax^{\frac{1}{2}} + Bx^{1} + Cx^{\frac{3}{2}} + Dx^{2} + Ex^{\frac{5}{2}} +$ &c. Sequitur operatio.

$$
\begin{array}{r|l}
+aa = & +aa \\
-axy^{2} = & -\frac{1}{4}aA^{2} - aABx^{\frac{1}{2}} \left.\begin{array}{l} -\frac{3}{2}aAC \\ -aBB \end{array}\right\} x \left.\begin{array}{l} -2aAD \\ -2aBC \end{array}\right\} x^{\frac{3}{2}} \\
 & \left.\begin{array}{l} -\frac{5}{2}aAE \\ -4aBD \\ -\frac{2}{4}aCC \end{array}\right\} x^{2} - \&c. \\
+yy = & \quad * \qquad * \qquad A^{2}x \qquad + - ABx^{\frac{3}{2}} \\
 & \left.\begin{array}{l} +2AC \\ +BB \end{array}\right\} x^{2} + \&c. \\
-3x^{2}y = & \quad * \qquad * \qquad * \qquad -\frac{3}{2}Ax^{\frac{3}{2}} \\
 & -3Bx^{2} - \&c. \\
+2x^{2} = & \quad * \qquad * \qquad * \qquad * \\
 & +2x^{2}
\end{array}
$$

$$
\text{Unde}\begin{cases}
A = +2\sqrt{a},\ B = 0,\ C = +\dfrac{4\sqrt{a}}{3a},\ D = -\dfrac{3}{4a}, \\
E = +\dfrac{2\sqrt{a}}{3aa},\ \&c. \\
A = -2\sqrt{a},\ B = 0,\ C = -\dfrac{4\sqrt{a}}{3a},\ D = -\dfrac{3}{4a}, \\
E = -\dfrac{2\sqrt{a}}{3aa},\ \&c.
\end{cases}
$$

$$
\text{Ergo } y = \begin{cases}
+2\sqrt{ax} + \dfrac{4x\sqrt{ax}}{3a} - \dfrac{3x^{2}}{4a} + \dfrac{2x^{2}\sqrt{ax}}{3aa}\ \&c. \\
-2\sqrt{ax} - \dfrac{4x\sqrt{ax}}{3a} - \dfrac{3x^{2}}{4a} - \dfrac{2x^{2}\sqrt{ax}}{3aa}\ \&c.
\end{cases}
$$

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Sed ut verum quod eſt confiteamur, hæc methodus D. *Taylor* inveniendi formam ſeriei non eſt generalis. Imo impoſſibile eſt univerſaliter determinare formam ſeriei ex datâ formâ æquationis; pendet enim forma ſeriei tam ex coefficientibus, quam ex exponentibus indeterminatarum in æquatione. Sint enim duæ æquationes $y^3 - 3ay^2 + 3a^2y - a^3 + x^2y = 0$, $y^3 - ay^2 + a^2y - a^3 + x^2y = 0$ eædem omnino quoad formam, in priore eſt $y = a - a^{\frac{1}{3}}x^{\frac{2}{3}} + \frac{x^{\frac{4}{3}}}{3a^{\frac{1}{3}}}$ &c. in poſteriore eſt $y = a + \frac{x^2}{2a} - \frac{x^4}{2a^3}$, &c. Et juxta Regulam D. *Taylor*, utraque ſeries habere debet eandem formam, quam habet poſterior. Fallit dicta Regula quotieſcunque coincidunt duo vel plures valores termini primi ſeriei: hoc eſt, quando ordinata prima tangit curvam vel tranſit per punctum ejus duplex: vel quando ordinata ad diſtantiam infinitam tranſit per plura curvæ puncta infinite propinqua ad ſe invicem. Inveni autem ſequentem regulam pro inveniendâ formâ ſeriei quantum hactenus conſtitit nunquam fallere, ſed illam eſſe ubique veram affirmare non audeo, propterea quod in eam caſu tantum incidi, (obſervando ſcilicet plurimas ſeries diverſas) & ejus demonſtrationem poſtea fruſtra quæſivi.

Methodus determinandi formam Seriei.

INveniatur index primi termini, vocetur is n, in æquatione pro y, y, y, &c. ſcribe x^n, x^{n-1}, x^{n-2} &c.

re-

reſpective, adeo ut termini reſultantes componantur ex x & datis quantitatibus: ſit m maximus communis diviſor indicum terminorum ſic reſultantium, p numerus valorum primi termini qui inter ſe æquantur, & $r = \frac{m}{p}$, atque forma ſeriei hæc erit $y = Ax^n + Bx^{n+r} + Cx^{n+2r} + \&c.$

Coincidit hæc regula cum regula D. *Taylor* quando p eſt unitas, hoc eſt, quando primus terminus non habet plures valores æquales.

Exemplum Quartum.

EX æquatione $y^3 - 2xy^2 + x^2y - a^3 = 0$, extrahenda ſit radix y in ſerie eo citius convergente quo major eſt x. Invenies ope rectanguli primum terminum eſſe x, quem ſcribe pro y & æquatio evadet $x^3 - 2x^3 + x^3 - a^3 = 0$. Indices terminorum ſunt 3, 3, 3, 0, quorum maximus communis diviſor eſt 3. & æquatio $y^3 - 2xy^2 + x^2y = 0$ quæ dat primum terminum habet duas radices inter ſe æquales; igitur eſt $m = 3$, $p = 2$, & $\frac{m}{p} = \frac{3}{4} = r$, unde forma ſeriei hæc eſt $y = Ax + Bx^{-\frac{1}{2}} + Cx^{-2} + Dx^{-\frac{7}{2}} + \&c.$

Eſtque

Eſtque operatio ut ſequitur.

$$
\begin{aligned}
y^3 = {} & A^3x^3 + 3A^2Bx^{+\frac{3}{2}} \left.\begin{matrix} +3A^2C \\ +3AB^2 \end{matrix}\right\} x^0 \left.\begin{matrix} +3A^2D \\ +6ABC \\ +B^3 \end{matrix}\right\} x^{-\frac{3}{2}} \\
& \left.\begin{matrix} +3A^2E \\ +6ABD \\ +3AC^2 \\ +3B^2C \end{matrix}\right\} x^{-3} + \&c. \\
-2xy^2 = {} & -2A^3x^3 - 4ABx^{\frac{3}{2}} \left.\begin{matrix} -4AC \\ -2B^2 \end{matrix}\right\} x^0 \left.\begin{matrix} -4AD \\ -4BC \end{matrix}\right\} x^{-\frac{3}{2}} \\
& \left.\begin{matrix} -4AE \\ -4BD \\ -2CC \end{matrix}\right\} x^{-3} - \&c. \\
+x^2y = {} & Ax^3 + Bx^{\frac{3}{2}} + Cx^0 + Dx^{-\frac{3}{2}} \\
& + Ex^{-3} + \&c. \\
-a^3 = {} & -a^3x^0
\end{aligned}
$$

Primo igitur habemus æquationem, $A^3 - 2A^2 + A = 0$, unde eſt $A = +1$, $A = +1$, & $A = 0$. Secundo eſt $3A^2B - 4AB + B = 0$, vel $3B - 4B + B = 0$, ſed inde nihil colligitur. Tertio eſt $3A^2C + 3AB^2 - 4AC - 2B^2 + C - a^3 = 0$, vel $3AB^2 - 2B^2 - a^3 = 0$, unde eſt $B = \pm a^{\frac{3}{2}}$, Quarto eſt $3A^2D + 6ABC + B^3 - 4AD - 4BC + D = 0$, id eſt, $6ABC + B^3 - 4BC = 0$, ergo eſt $C = -\frac{a^3}{2}$, Et quinto invenietur $D = \frac{7}{8}a^{\frac{3}{2}}$ quando pro B ſcribitur $+a^{\frac{3}{2}}$, Sed erit $= -\frac{7}{8}a^{\frac{3}{2}}$, quando pro B ſcribitur $-a^{\frac{3}{2}}$. Adeoque eſt radix

$y =$

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$$y = \begin{cases} x + \dfrac{aa}{\sqrt{ax}} - \dfrac{a^3}{2x^2} + \dfrac{7a^4}{8x^2\sqrt{ax}} \; \&c. \\ x - \dfrac{aa}{\sqrt{ax}} - \dfrac{a^3}{2x^2} - \dfrac{7a^4}{8x^2\sqrt{ax}} \; \&c. \end{cases}$$

Hic in determinatione Coefficientium obſervandum eſt quod ex ſecundâ æquatione $3A^2B - 4AB + B = 0$ terminus ſecundus B non reperiebatur. Idem ſemper notandum eſt quotieſcunque terminus primus habet plures valores inter ſe æquales. Nam ſi termini primi omnes valores ſunt inter ſe diverſi, terminus ſecundus & reliqui omnes in infinitum habebunt niſi unicum valorem & per diviſionem ſemper prodibunt. At ſi terminus primus habet plures valores inter ſe æquales, tot diverſos valores neceſſario habebit terminus ſecundus; qui itaque per diviſionem inveniri nequit, ſed radix erit æquationis tot dimenſionum quot ipſe habet valores. Unde in comparatione Coefficientium ſecundus terminus B ex æquatione ſecundâ non ſemper invenitur: Sed ponendo Coefficientes terminorum homologorum æquales nihil, membra omnia ſe mutuo ſemper deſtruent uſque dum pervenietur ad terminum in cujus coefficiente reperitur terminus ſecundus tot dimenſionum quot ipſe habet valores. Sed hæc omnia experientiâ multo melius quam verbis patebunt.

Conſiderando Curvarum Tangentes, Curvaturam, Variationem Curvaturæ, Variationem, Vari-

Variationis, &c. quæ determinari ſolent Fluxionum ope, determinari etiam poſſe ope terminorum ſeriei, ut innuit D. *Neutonus* ad Prop. 10. Lib. 2. *Principiorum*; ſtatim cognovi Incrementa prima, ſecunda, tertia, &c. relationem quandam habere ad ſeriei terminos reſpectivos, adeoque terminos illos determinari ex Fluxionibus. Ideo quærebam relationem illam & tandem inveni ut ſequitur.

PROP. III. THEOR.

SIT $y = A + Bx^r + Cx^{2r} + Dx^{3r} + Ex^{4r} + \&c.$ erit

$$B = \frac{\dot{y}}{1.r\dot{x}},\quad C = \frac{\ddot{y}}{1.2.r^2\dot{x}^2},\quad D = \frac{\dddot{y}}{1.2.3.r^3\dot{x}^3},$$

$$E = \frac{\ddddot{y}}{1.2.3.4.r^4\dot{x}^4}$$ &c. adeoque eſt $y = A + \frac{x\dot{y}}{1.r\dot{x}}$

$$+ \frac{x^2\ddot{y}}{1.2.r^2\dot{x}^2} + \frac{x^3\dddot{y}}{1.2.3.r^3\dot{x}^3} + \&c.$$

Demonſtratio.

Fluat x^r uniformiter, & ſit ejus Fluxio $rx^{r-1}\dot{x} = 1$, vel $\dot{x} = \frac{1}{r}x^{1-r}$. Et erit $\dot{y} = rBx^{r-1}\dot{x} + 2rCx^{2r-1}\dot{x} + 3rDx^{3r-1}\dot{x} + 4rEx^{4r-1}\dot{x} + \&c.$ ubi ſi pro $\dot{x}$ ponas ejus valorem $\frac{1}{r}x^{1-r}$ erit $\dot{y} = B + 2Cx^r + 3Dx^{2r}$

+4E

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$+ 4Ex^{3r} +$ &c. Et inde $\dot{y} = 2rCx^{r-1}\dot{x} + 6rDx^{2r-1}\dot{x} + 12rEx^{3r-1}\dot{x} +$ &c. vel ponendo pro $\dot{x}$, $\frac{1}{r}x^{1-r}$, $\ddot{y} = 2C + 6Dx^r + 12Ex^{2r} +$ &c. unde $\dddot{y} = 6rDx^{r-1}\dot{x} + 24rEx^{2r-1}\dot{x} +$ &c. id eſt, $\dddot{y} = 6D + 24Ex^r +$ &c. adeoque $\ddddot{y} = 24rEx^{r-1}\dot{x} +$ &c. $= 24E +$ &c. Jam ſit x infinite magna vel infinite parva, prout r eſt numerus negativus aut affirmativus, & erit accurate $\dot{y} = B$, $\ddot{y} = 2C$, $\dddot{y} = 6D$, $\ddddot{y} = 24E$, &c. ergo viciſſim eſt $B = \dot{y}$, $C = \frac{1}{2}\ddot{y}$, $D = \frac{1}{6}\dddot{y}$, $E = \frac{1}{24}\ddddot{y}$, &c. Hiſce valoribus ipſorum B, C, D, E, &c. ſubſtitutis orietur $y = A + x^r\dot{y} + \frac{1}{2}x^{2r}\ddot{y} + \frac{1}{6}x^{3r}\dddot{y} + \frac{1}{24}x^{4r}\ddddot{y} +$ &c. Sed erat $\dot{x} = \frac{1}{r}x^{1-r}$, ideoque eſt $x^r = \frac{x}{\dot{x}r}$ quem ipſius x^r valorem ſubſtitue ut termini evadant homogenei, & prodibit $y = A + \frac{x\dot{y}}{1.r\dot{x}} + \frac{x^2\ddot{y}}{1.2.r^2\dot{x}^2} + \frac{x^3\dddot{y}}{1.2.3.r^3\dot{x}^3}$

$+ \frac{x^4\ddddot{y}}{1.2.3.4.r^4\dot{x}^4} +$ &c. Q. E. D.

Coroll. Sit $y = Ax^n + Bx^{n+r} + Cx^{n+2r} + Dx^{n+3r}$ $+$ &c. id eſt $y = x^n \times \overline{A + Bx^r + Cx^{2r} + Dx^{3r} + \&c.}$ pone $y = vx^n$, & erit $v = A + Bx^r + Cx^{2r} + Dx^{3r} +$ &c. Uunde per hanc Propoſitionem eſt $v = A + \frac{x\dot{v}}{1.r\dot{x}}$

$+\frac{x^2\ddot{v}}{1.2.r^2x^2}+\frac{x^3\dddot{v}}{1.2.3.r^3x^3}+$ &c. Ergo eſt $\dot{y}=x^n\times\overline{A}$

$+\frac{x\dot{v}}{1.rx}+\frac{x^2\ddot{v}}{1.2.r^2x^2}+\frac{x^3\dddot{v}}{1.2.3.r^3x^3}+$ &c. Ut habeantur Fluxiones ipſius v, pro y in æquatione ſubſtitue vx^n, & orietur æquatio nova relationem inter x & v definiens, ex qua invenies fluxiones illas.

Exemplum Primum.

Ex æquatione $y^4-a^3y=ax^3-a^3x$ extrahenda ſit radix y. Invenio formam ſeriei fore $A+Bx+Cx^2+Dx^3+$ &c. adeoque eſt $r=1$. Et ideo fluit x uniformiter, unde per methodum Fluxionum directam eſt $4y^3\dot{y}-a^3\dot{y}=3ax^2-a^3$, $4y^3\ddot{y}-a^3\ddot{y}=6ax-12y^2\dot{y}^2$, $4y^3\dddot{y}-a^3\dddot{y}=6a-24y\dot{y}^3-36y^2\dot{y}\ddot{y}$, &c. unde $\dot{y}=\frac{3ax^2-a^3}{4y^3-a^3}$, ubi pro x ſcribe o, & pro y a, ejus valorem cum x eſt o, & erit $\dot{y}=\frac{-a^3}{4a^3-a^3}=-\frac{1}{3}$:

$$\ddot{y}=\frac{6ax-12y^2\dot{y}^2}{4y^3-a^3}=\frac{-12\times a^2\times\frac{1}{9}}{3a^3}=\frac{-4}{9a};$$

$$\dddot{y}=\frac{6a-24y\dot{y}^3-36y^2\dot{y}\ddot{y}}{4y^3-a^3}=\frac{6a+\frac{24}{27}a-\frac{36\times4a}{27}}{3a^3}$$

$=\frac{50}{27a^2}$. In ſerie generali hoſce valores ſubſtitue, unitatem pro r & a pro A, atque proveniet $y=a-\frac{1}{3}x-\frac{2x^2}{9a}+\frac{25x^3}{81a^2}$ &c.

Exem-

Exemplum Secundum.

ÆQuationis $y^3 + a^2y + x^2y - 2a^3 = 0$: quæratur radix y. Invenio formam ſeriei eſſe $A + Bx^2 + Cx^4 + \&c.$ igitur eſt $r = 2$, & x^2 fluit uniformiter, id eſt, $2x\dot{x} = 1$, & $\dot{x} = \frac{1}{2x}$. Capiendo fluxiones erit $3y^2\dot{y} + a^2\dot{y} + x^2\dot{y} = -2x\dot{x}y = -y$; $3y^2\ddot{y} + a^2\ddot{y} + x^2\ddot{y} = -2\dot{y} - 6y\dot{y}^2$, $3y^2\dddot{y} + a^2\dddot{y} + x^2\dddot{y} = -3\ddot{y} - 18y\dot{y}\ddot{y} - 6\dot{y}^3$. Unde $\dot{y} = \frac{-y}{3y^2 + a^2}$, neglecto termino $x^2\dot{y}$, quoniam ſupponitur x evaneſcere; pro y pone a ejus valorem cum eſt $x = 0$, & erit $\dot{y} = -\frac{1}{4a}$, $\ddot{y} = \frac{-2\dot{y} - 6y\dot{y}^2}{3y^2 + a^2}$

$$= \frac{+\frac{2}{4a} - 6a \times \frac{1}{16a^2}}{4a^2} = \frac{1}{32a^3},\ \dddot{y} = \frac{-3\ddot{y} - 18y\dot{y}\ddot{y} - 6\dot{y}^3}{3y^2 + a^2}$$

$$= \frac{\frac{-3}{32a^3} - 18a \times -\frac{1}{4a} \times \frac{1}{32a^3} + \frac{6}{64a^3}}{4a^2} = \frac{-9}{256a^5}.$$

In ſerie generali pro A ſubſtitue a, & 2 pro r, atque proveniet $y = a - \frac{x^2}{4a} + \frac{x^4}{64a^3} - \frac{3x^6}{512a^5}$ &c.

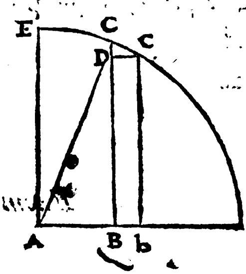
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Exemplum Tertium.

ELevanda ſit binomium $a+x$ ad poteſtatem indeterminatam cujus index eſt n. Pone $y = \overline{a+x}|^n$; unde cum eſt $x = 0$ evadit $y = a^n$, & inde in ſerie generali eſt $A = a^n$. Forma ſeriei hæc eſt $A + Bx + Cx^2 + Dx^3 +$ &c. ergo eſt $r = x = 1$. Capiendo fluxiones erit $a\dot{y} + x\dot{y} = ny$, $a\ddot{y} + x\ddot{y} = \overline{n-1} \times \dot{y}$, $a\dddot{y} + x\dddot{y} = \overline{n-2} \times \ddot{y}$, $a\ddddot{y} + x\ddddot{y} = \overline{n-3} \times \dddot{y}$, &c. Evaneſcat x, & erit $\dot{y} = na^{n-1}$, $\ddot{y} = n \times \overline{n-1} \times a^{n-2}$, $\dddot{y} = n \times \overline{n-1} \times \overline{n-2} \times a^{n-3}$, $\ddddot{y} = n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times a^{n-4}$, &c. In ſerie generali hoſce valores ſubſtitue a^n pro A & unitatem pro r atque prodibit $y (= \overline{a+x}|^n) = a^n + na^{n-1}x + \frac{n}{1} \times \frac{n-1}{2} a^{n-2}x^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3}x^3$ &c. Hoc modo patet quanta facilitate demonſtratur Theorema D. *Neutoni*.

Exemplum Quartum.

EX dato arcu EC quæratur ejus Coſinus BC. Sit $EC = x$, $BC = y$, $Cc = \dot{x}$, $DC = \dot{y}$; Erit $AB = \sqrt{1-y^2}$, exiſtente radio $AE = 1$. Propter ſimilia triangula ABC, CDc, $AC : AB :: Cc : CD$, id eſt, $1 : \sqrt{1-y^2} :: \dot{x} : \dot{y}$, unde erit $\dot{y}^2 = \dot{x}^2 - \dot{x}^2 y^2$. Forma ſeriei



riei hæc eſt $A + Bx^2 + Cx^4 + Dx^6 +$ &c. quæ continentur in hâc formâ, $A + Bx + Cx^2 + Dx^3 + Ex^4$ + &c. igitur poteſt x vel x^2 fluere uniformiter quum quæruntur fluxiones ipſius y. Fluat x uniformiter & ſit $\dot{x} = 1$, atque æquatio evadet $\dot{y}^2 = 1 - y^2$, unde $2\dot{y}\ddot{y} = -2y\dot{y}$ vel $\ddot{y} = -y$, & inde $\dddot{y} = -\dot{y}$, $\ddddot{y} = -\ddot{y}$, $\overset{.....}{y} = -\dddot{y}$, &c. Quoniam exiſtente arcu $EC = 0$, fit $y = AE = 1$, pro y ſubſtitue unitatem, & erit $\dot{y} = 0$, $\ddot{y} = -1$, $\dddot{y} = 0$, $\ddddot{y} = 1$, $\overset{.....}{y} = 0$, &c. In ſerie generali pro hiſce fluxionibus hoſce valores ſubſtitue, unitatem pro A & unitatem pro r, atque proveniet $y = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} +$ &c. Et eodem modo invenire licet ſinum AB ex dato ejus arcu EC.

Ex hoc Exemplo conſtat operationem admodum alleviari ſupponendo x & non x^2 fluere uniformiter. Idem alibi notandum eſt, nam r (quæcunque ſit forma ſeriei) ad libitum fere ſumi poteſt. Ut ſi forma ſeriei quæſitæ eſſet $Ax^{\frac{3}{2}} + Bx^{\frac{7}{2}} + Dx^{\frac{11}{2}} + Ex^{\frac{15}{2}} +$ &c. ubi differentia Exponentium eſt 2, non opus eſt ut ſit $r = 2$, ſed poteſt eſſe quilibet numerorum ſequentium $2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, &c.

Methodus hæc reducendi radices æquationum in ſeries convergentes, haud abſimilis eſt methodo aſſumendi ſeriem coefficientibus indeterminatis affectam; at plerumque minus operoſa, præſertim ſi deſiderentur

ſeriei

ſeriei termini tantum pauci initiales: hi ſufficiunt ad inveniendas Tangentes, Radios Curvaturæ Aſymptotos & ſimilia.

De Æquationum Reſolutione in numeris.

HÆC de æquationum reductione Literali dicta ſufficiant, de Numerali itaque adjicere pauca licet.

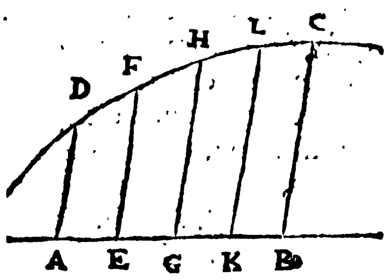
Omnis Æquationum Reductio, uti ſupra diximus, ex hoc pendet, ut primo aſſumatur quantitas radici quæſitæ æqualis proxime, & poſtea ut valor ille aſſumptus corrigatur. Exempli gratiâ, ſit æquatio Cubica $y^3 - 21y - 16 = 0$, cujus radix quæritur; & tentando, vel per Conſtructionem Geometricam invenio eſſe 5 numerum prope verum, itaque pro y ſubſtituo $p + 5$, & provenit æquatio nova $p^3 + 15p^2 + 54p + 4 = 0$; jam quoniam eſt y proxime æqualis 5, erit p quantitas admodum parva, & per conſequens ejus poteſtates altiores, erunt ipſo multo minores; neglectis igitur terminis minoribus p^3, $15p^2$ erit fere $54p + 4 = 0$; vel $p = -\frac{4}{54} = -,074$, ergo $p + 5 = y = 4,926$ proxime. Si quæritur radix magis accurata, pro p ponendum eſt $q - ,074$, atque operationem perficiendo invenies plures figuras radici jam inventæ adnectendas.

Hæc methodus ea eſt quam invenit & in Analyſi ſuâ tradidit D. *Neutonus*. Sunt & aliæ methodi idem perficiendi, qualis eſt ea *J. Ralphſon* & Cl. *Hallei*, ſed ex eâ jam traditâ omnes pendent & facile fluunt.

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Hoc modo reducuntur æquationes omnes vulgares, in quibus ſcilicet exiſtit unica tantum quantitas ignota: verum etiam æquationes hujuſmodi $y' + 3' - 7y = 0$ ſimili modo poſſunt reduci. Sed in æquationes iſtiuſmodi, quarum uſus nondum pene innotuit, tempus impendere jam non vacat. Æquationes fluxionales, quoniam in iis ſemper reperiuntur plures quantitates incognitæ utpote radix extrahenda cum fluxionibus ſuis, dicto modo ſunt irreducibiles. Ut ergo earum radices in numeris haberi poſſunt, ad ſeries confugiendum eſt. Ex ſeriebus enim, quotieſcunque ſat celeriter convergunt, expedite inveniuntur radices æquationum. Si quando non convergunt, tantum opus eſt (monente ipſo *Neutono* ſub finem *Analyſeos*) ut x, ſcilicet quantitas ex cujus poteſtatibus conficitur ſeries, aliquoties adhuc minor ſupponatur; adeoque radix non unica ſed pluribus ſeriebus habenda eſt. Quomodo vero hoc fit, ſtatim erit manifeſtum.

Sit curva *DC* cujus Abſciſſa $AB = x$, Ordinata $BC = y$. Et quæratur y vel *BC* in numeris quando Abſciſſa evadit *AB*. Reducatur Ordinata in ſeriem, & ſi ſeries illa ſat cito convergat dabitur *BC*:



at ſi non convergat, eouſque minui intelligatur

x ut

x ut tandem celeriter convergat ſeries; in illâ magnitudine ſit $x = AE$, & ſeries dabit in numeris Ordinatam *EF*. Jam ſumo novam Abſciſſam $EB = z$, & quæro relationem inter *z* & *y*, ex quâ invenietur valor ipſius *y* in ſerie ex poteſtatibus ipſius *z* confecta. Si ſeries illa ſat cito convergat, quum eſt $z = EB$, habemus quæſitum; at ſi non convergat, eodem modo minui jam intelligatur *z* quo prius *x*, ut convergat ſeries; ſit in illo caſu $z = EG$, & dabitur Ordinata *GH*. Rurſus ſit Abſciſſa tertia $GB = v$, & quæratur relatio inter *v* & *y*, quâ inventâ reducatur *y* in ſeriem ex dignitatibus ipſius *v* confectam; quæ, ſi non aduc celeriter convergit, minuatur *v* & ſit ea $= GK$, quæ *GK* ſit tam parva ut ſeries ad libitum convergat, & dabitur Ordinata *KL*. Sumo quarto Abſciſſam $KB = u$, & quæro relationem inter *u* & *y*, ex quâ invenio valorem ordinatæ *y* in ſerie, quæ ſi cito convergat habemus ipſam *BC* quam quærimus. At ſi non convergit eodem modo continue procedere licet quo prius: atque ex tali proceſſu Abſciſſa ultima *KB* tandem evadet minor datâ quâvis quantitate; ideoque ſeries ultima ad libitum converget.

Hic notandum eſt quod eſt *AD* primus terminus ſeriei, quando *AE* eſt Abſciſſa & *EF* Ordinata, *EF* eſt primus terminus quando *EG* eſt Abſciſſa & *GH* Ordinata, *GH* eſt primus terminus quando *GK* eſt Abſciſſa & *KL* Ordinata, *KL* eſt primus terminus ſeriei quando *KB* eſt Abſciſſa & *BC* Ordinata. Ideoque priuſquam ha-

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haberi poteſt Ordinata quævis ſubſequens, neceſſario habendæ ſunt omnes antecendentes, & ergo antequam ordinata *BC* inveniri poſſit, inveniendæ ſunt omnes aliæ. Adjiciendum jam eſſet Exemplum unum aut alterum, ſed methodum hanc per ſe ſatis patentem ducens, tædium calculi evitare inſtitui.

Subnecterem jam quædam de *Præparatione* Æquationum: nam aliquæ Æquationes præpatione indigent antequam earum radices erui poſſunt. In æquationibus Lineas Rationales deſignantibus nullam novi difficultatem ex hactenus dictis facile non tollendam: non item in Æquationibus Fluxiones involventibus. Hæ aliquando præparantur mutando Ordinatam, alias Abſciſſam, aliquando utramque: Sed ſunt æquationes quibus nulla ſufficit præparatio, quantum mihi conſtare potuit: ego de re dubiâ tractare non ſuſcipio. Certe illi quicunque hanc materiam aggreditur prodibunt calculi quorum onus ægre ſuſtinendum eſt.

PROP. IV. THEOR.

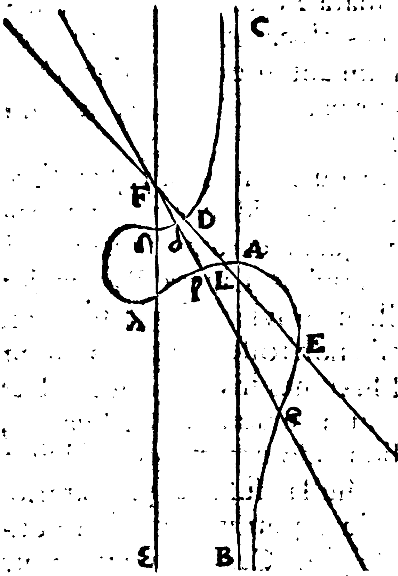
Aſymptotos recta decuſſare poteſt curvam in totidem punctis, demptis duobus, quot curva eſt dimenſionum & nunquam pluribus.

LInea quævis ſecari poteſt à rectâ in tot punctis quot ipſa eſt dimenſionum & nunquam pluribus, quoniam æquatio tot habere poteſt radices quot ipſa eſt dimenſionum & non plures. Sit jam v.g. linea tertii ordinis Aſymptoton

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habens *BAC*. Duc rectam quamvis *FDLE* ſecantem curvam in tribus punctis *D*, *L*, *E*. In hac recta ſit punctum quodlibet *F*, circa quod tanquam polum gyretur recta *DE*, per ſitum *Fdle* tandem in ſitum *F* δ λ ε perveniens; ubi eſt Aſymptoto parallela: & *E* unum interſectionis punctum abiit in infinitum: ideoque in illâ poſitione recta δε occurrit curvæ in duobus tantum punctis δ, λ. Moveatur recta δλ motu parallelo donec tandem cum Aſymptoto *BC* coëat, & in illo caſu punctum δ etiam abibit in infinitum: reſtat igitur unicum punctum *A* in quo Aſymptotos curvam decuſſare poteſt. Et ſimiliter in ullo alio caſu oſtendetur duo interſectionis puncta abire in infinitum, hoc eſt, evaneſcere & nullibi reperiri. Proinde reſtabit numerus punctorum, in quibus Aſymptotos decuſſare poteſt curvam, æqualis numero dimenſionum curvæ dempto binario Q. E. D.



Coroll. 1. Linea ſecundi ordinis, id eſt, ſectio Coni ejus Aſymptoton non omnino decuſſat,

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Linea tertii ordinis ejus Aſymptoton decuſſare poteſt in unico tantum puncto, Linea quarti ordinis in duobus & nunquam pluribus. Et ſic in aliis.

Coroll. 2. Linea ſecundi vel tertii ordinis Aſymptoton non tanget, Linea quarti vel quinti ordinis ejus Aſymptoton tangere poteſt in unico puncto. Et ſic porro. Liquet hoc Corollarium exinde quod punctum contactus conflatur ex pluribus interſectionum punctis in unum coeuntibus.

Coroll. 3. Si Linea quarti ordinis tangat ejus Aſymptoton, radius Curvaturæ in puncto contactus ſemper erit finitus: nam punctum illud contactus conflabitur ex duobus tantum interſectionum punctis.

Coroll. 4. Si duo crura Aſymptoton aliquam adjacentia, jaceant ad eaſdem ejus partes, vel ad contrarias & ſimul ad eandem plagam protenſa, tria ad minimum interſectionis puncta abeunt in infinitum.

Coroll. 5. Hinc colligimus maximum numerum Aſymptoton parallelarum quas Linea quævis habere poteſt, æqualem eſſe numero ejus dimenſionum demptâ unitate.

Coroll. 6. Et ſi curva habeat tot Aſymptotos parallelas, demptâ unitate quot ipſa eſt dimenſionum, ea earum nullam ſecare poteſt.

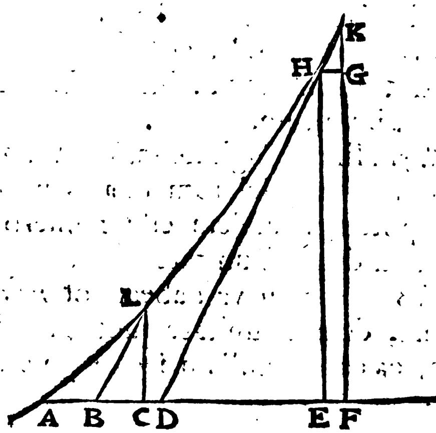
E 2 PROP.

PROP. V. THEOR.

Si Ordinata curvæ parallela ſit Tangenti ad punctum infinite diſtans, Ordinata illa in æquatione curvam definiente non aſcendet ad tot dimenſiones quot eſt curva.

SIT *ALHK* crus curvæ infinitum, Abſciſſa *AC* = *x*, Ordinata *CL* = *y*, *DHK* tangens ad *H* punctum infinite diſtans, cui parallela ſit Ordinata *BL* = *v*, eique ſit correſpondens Abſciſſa *AB* = *z*. Dico in æquatione ad curvam quod *v* non aſcendet ad tot dimenſiones quot eſt curva.

Per *H* duc Ordinatam *HE*, cui praralleła *KGF*, exiſtente *EF* quam minima, ſit porro *GH*



Abſciſſæ parallela. Concipiatur punctum *L* abire in infinitum, adeo ut coëant puncta *L*, *H*; *B*, *D*;

B, *D*; *C*, *E*; Et in illo caſu erit $AE = x$, $EH = y$, $GH = x$, $GK = y$, $AD = z$, $DH = v$. Reducantur y in ſeriem hujuſmodi $Ax + Bx^{1-n} + Cx^{1-2n} + Dx^{1-3n} +$ &c. eo citius convergentem quo major eſt x. Index ipſius x in primo termino ſeriei neceſſario erit unitas, quoniam ultima tagens *DH* datam ad Abſciſſam *AB* ſupponitur habere inclinationem. Erit $\dot{y} = A\dot{x} + \overline{1-n} \times Bx^{-n}\dot{x} + \overline{1-2n} \times Cx^{-2n}\dot{x} +$ &c. Unde eſt $\dot{x} : \dot{y} :: 1 : A + \overline{1-n} \times Bx^{-n} + \overline{1-2n} \times Cx^{-2n}$ + &c. vel ob ſimilia triangula *DEH*, *HGK* $1 : A + \overline{1-n} \times Bx^{-1} + \overline{1-2n} \times Cx^{-2n} +$ &c. $:: DE : y$; ideoque invenietur $DE = x + \frac{nB}{A}x^{1-n} +$ &c. ponendo pro y ejus valorem in ſerie. Et $AD = AE - DE = -\frac{nB}{A}x^{1-n} +$ &c. Sit jam x infinite magna, & erit accurate $y = EH = Ax$, $AD = -\frac{nB}{A}x^{1-n}$, quumque per naturam ſeriei ſit n numerus affirmativus, erit *AD* infinite minor *EH*, & etiam infinite minor *DH* quæ ad *EH* datam habet rationem: hoc eſt z infinite minor v, igitur in æquatione v non erit tot dimenſionum quot eſt z, adeoque nec tot quot eſt curva. Q. E. D.

Coroll.

Coroll. 1. Igitur æquatio $x = a$ deſignat rectam, ubi ordinata y nullius eſt dimenſionis, $\overline{x + a}\, x y = b x^2 + c x + d$ omnes ſecundi ordinis quæ pergunt in infinitum, $\overline{x + a}\, x y y = \overline{b x^2 + c x + d}\, x y + e x^3 + f x^2 + g x + h$ omnes tertii ordinis: & ſic in reliquis.

Coroll. 2. Si exiſtente Abſciſsâ curvæ infinite magnâ, Ordinata ſit adhuc infinite major, ea parallela erit Tangenti Cruris Parabolici ad diſtantiam infinitam.

Coroll. 3. Sin Ordinata ſit infinite magna, quum Abſciſſa non eſt infinite magna, ea parallela erit Aſymptoto cruris Hyperbolici.

Coroll. 4. Conſtat methodus determinandi poſitionem Tangentium curvarum ad diſtantiam infinitam, id eſt poſitionem Ordinatarum quæ in æquatione non aſcendent ad tot dimenſiones quot eſt curva, ſcilicet ex ratione quam ad invicem habent x, y in ultimâ earum magnitudine: quæ ratio ſemper invenitur ope ſerierum convergentium.

Scholion.

POteram hanc Propoſitionem eodem modo demonſtraſſe quo priorem. Nam in æquatione tot ſemper peribunt Ordinatæ dimenſiones, quot interſectionis ejus puncta abeunt in infinitum. Et ſi quando in aliquâ curvâ Ordinata non eſt tot dimenſionum quot eſt curva, ſemper comcipiendum eſt puncta quædam interſectionis abire in infinitum: imo hoc in ipſis

ſis Ovalibus obtinet; nam concipiendum eſt eas habere puncta duplicia imaginaria ad diſtantiam infinitam. Finge enim curvam habere tot Aſymptotos quot habet dimenſiones, quarum nullæ duæ ſunt inter ſe parallelæ: atque Ordinatæ illis æquidiſtantes tot erunt dimenſionum quot eſt Curva, demptâ unitate. Concipe Aſymptotos plures motu angulari latas, donec tandem evadant parallelæ, & ordinatæ omnibus illis parallelæ perdent tot dimenſiones quot ſunt Aſymptoti æquidiſtantes. Quod ſi forte æquatio Aſymptotos duas vel plures determinans evadat impoſſibilis, evaneſcent Aſymptoti illæ cum earum cruribus; at Ordinatæ non erunt plurium dimenſionum quam antea erant: Atque hinc redditur ratio, quomodo in Ovalibus aliiſque curvis, Ordinatarum nulli tangenti ultimæ parallelarum evaneſcunt dimenſiones: ſequitur vero in illis caſibus dimenſionum evaneſcentium numerum ſemper eſſe parem, quoniam radicum impoſſibilium numerus eſt par. Si forte evenit, quod Ordinatæ unicæ Aſymptoto parallelæ perdunt plures dimenſiones, & nulla interim comparet æquatio, per cujus radicum impoſſibilitatem Aſymptoti reliquæ evanuerunt; tum concipe plures Aſymptotos in unam coire. In illis caſibus Aſymtotos ſemper habet crura plura ſolito ad eaſdem plagas extenſa. Numerus vero Aſymptoton coeuntium quas curva quævis habere poteſt, æqualis eſt numero Aſymptoton parallelarum quas habere poteſt; priuſquam enim coëunt evadunt parallelæ.

PROP.

PROP. VI. PROBL.

Invenire Aſymptotos Curvarum.

Ex datâ æquatione ad curvam reducatur Ordinata y in ſeriem hujuſmodi $y = Ax^n + Bx^{n-r} + Cx^{n-2r} +$ &c. eo citius covergentem quo major eſt Abſciſſa x: ſume Ordinatam novam z æqualem terminis omnibus hujus ſeriei initialibus, qui augendo x non minuuntur: & Ordinatarum y, z differentia augendo x continue diminuetur, atque Ordinatæ ipſæ ad æqualitatem magis magiſque tendent: Unde Linearum Abſciſſam communem x & Ordinatas y, z habentium Crura tanto propius ad ſe invicem continue accedunt, quanto magis producuntur, & tandem coincident: atque adeo z eſt Ordinata Aſymptoti, quæ dabitur ex æquatione eam definiente. Q. E. I.

Coroll. 1. Igitur infinita ſunt crurum genera, quo ſimplicior eſt ordinata z, eo ſimplicius erit crus: quod ſi ſit z ordinata rectæ, crus erit Hyperbolicum.

Coroll. 2. Si primus terminus ſeriei, qui augendo x minuitur, ſit affirmativus, Aſymptotos jacet inter curvam & Abſciſſam; ſin minus curva jacet inter Aſymptoton & Abſciſſam. Nam terminus ille ſeriei evadit æqualis parti Ordinatæ inter crus & ejus Aſymptoton interceptæ, ubi eſt x infinite magna.

Coroll.

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Coroll. 3. Ordinatæ Ovalium reduci nequeunt in ſeries ad veritatem tanto magis accedentes quanto major eſt x. Nam ſi hoc fieri poſſit, Ordinata Ovalis eſſet quantitas realis cum Abſciſſa eſt infinite magna. Sed Ordinata Ovalis eſt imaginaria cum Abſciſſa eſt infinite magna. Unde liquet Methodus dignoſcendi Ovales.

Coroll. 4. Simili prorſus ratiocinio colligitur, quod ſi quando Ordinata curvæ evadere poſſit infinite magna, non item Abſciſſa; valor Ordinatæ habere nequit in ſerie eo citius convergente quo major eſt Abſciſſa.

Coroll. 5. Omnis Linea imparis cujuſque Ordinis pergit in infinitum. Etenim Linea ordinis imparis deſignatur æquatione imparium dimenſionum; adeoque ad minimum reperietur una ſeries eo citius convergens quo major eſt x, propterea quod æquatio imparium dimenſionum ad minimum unam habet radicem poſſibilem. Et ſeries iſtiuſmodi (per *Coroll.* 3.) ad Ovales non extenditur, ergo ad curvas quæ progrediuntur in infinitum.

Coroll. 6. Hinc etiam ſequitur, quod Ovalium tum Abſciſſæ tum Ordinatæ ſemper ſunt parium dimenſionum. Nam ſi eſſet alterutra imparium, curva (per *Coroll.* 5.) pergeret in infinitum.

Coroll. 7. Linea quævis tot habere poteſt Aſymptotos quot ipſa eſt dimenſionum & nunquam plures. Nam tot habere poteſt, quot habet radices æquatio quæ dat primum terminum ſeriei $Ax + Bx^{1-r} + Cx^{n-2r} +$ &c. id eſt quot curva

 eſt

eſt dimenſionum, ut conſtat ex ſerierum doctrinâ.

Coroll. 8. Si terminorum initialium plures valores coincidant, Aſymptoti plures in unam coibunt: & figura habebit crura plura ſolito ad eaſdem plagas extenſa: ut accidit Conchoidi Veterum, quæ habet quatuor crura ad unicam Aſymptoton jacentia.

Coroll 9. Si numerus dimenſionum curvæ ſit par, & numerus Aſymptoton impar; vel ſi numerus dimenſionum ſit impar & numerus Aſymptoton par, figura habebit duo crura ad unicam Aſymptoton jacentia quæ in plagas eaſdem in infinitum ſerpunt.

Exempla.

1. SEries $x - \frac{1}{3}a + \frac{aa}{3x}$ &c. quam (in *Exemplo* 1. *Prop.* 2.) invenimus pro radice æqutionis $y^3 - a^2y + axy - x^3 = 0$, indicat curvam illam habere Aſymptoton æquatione $z = x - \frac{1}{3}a$ deſignatam, adeoque ejus crura eſſe Hyperbolica. Et quia $\frac{aa}{3x}$, terminus primus ſeriei qui augendo x minuitur, eſt affirmativus, Aſymptotos jacet (per *Coroll.* 2. *Prop.* 6.) inter Curvam & Abſciſſam. Et quoniam unica tantum ſeries iſtiuſmodi obtineri poteſt, Figura non habet niſi duo crura ad unicam Aſymptoton rectam jacentia.

2. Se-

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2. Series illa $\frac{a^2}{x} - \frac{a^3}{2x^2} + \frac{a^4}{6x^3} -$ &c. quam (in *Ex.* 2. *Prop.* 2.) invenimus pro radice æquationis $yx^3 + ayxx + a^2xx - 2a^3x = 0$ indicat Abſciſſam eſſe Aſymptoton habentem duo crura ad diverſas ejus partes jacentia & in plagas oppoſitas extenſa.

3. Series $x - \frac{1}{3}a + \frac{a^3}{81x^2}$ &c. quæ eſt radix æquationis $y^3 + axy - x^3 = 0$, indicat curvam illam habere duo crura ad eaſdem ejuſdem Aſymptoti rectæ partes jacentia & in plagas oppoſitas protenſa.

4. Series illæ duæ $x + \frac{1}{2}a - \frac{a^2}{4x}$ &c. & $2x + a - \frac{2a^2}{7x}$, &c. quæ (in *Ex.* 4. *Prop.* 2.) prodiere pro radicibus æquationis $x^2y^2 - 3x^2xy + 2x^2x^2 - axy^2 + a^2x^2 = 0$ indicant illam æquationem deſignare curvam habentem duas Aſymptotos rectas, quarum quælibet habet duo crura ad diverſas ejus partes jacentia & in plagas oppoſitas protenſa.

5. Series duæ $x + \frac{aa}{\sqrt{ax}} - \frac{a^3}{2x^2}$ &c. $x - \frac{aa}{\sqrt{ax}} - \frac{a^3}{2x^4}$ &c. indicant duas Aſymptotos in unam coiiſſe; & illam habere duo crura ad diverſas ejus partes jacentia & in plagas eaſdem infinite progredientia. Adeoque Ordinatæ Aſymptoti illi parallelæ (per *Coroll.* 4. *Prop.* 4.) erunt unius tantum dimenſionis. Habet vero curva aliam Aſymptoton quæ ex alia ſerie dabitur.

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Scholion.

METhodus inveniendi Aſymptotos in hâc Propoſitione expoſita eſt maxime generalis; ſunt & aliæ methodi quamplures particulares inveniendi Aſymptotos rectas, quas tamen omnes comprehendit ea jam tradita. Conſiderari poteſt Aſymptotos recta ut tangens ad punctum curvæ infinite diſtans, & hoc modo reducitur Aſymptotôn doctrina ad doctrinam tangentium, vel conſiderari poſſunt Aſymptoti tanquam extremæ crurum partes indirectum productæ. Sed omnes hæ methodi præſupponunt, aliquo ſaltem modo, ſerierum doctrinam.

Vide Fig. Prop. 5. Invenienda ſit Aſymptotos curvæ *ALH*, quam definat æquatio $y^3 - axy + x^3 = 0$, ubi x & y eaſdem deſignant rectas quas in dictâ Propoſitione deſignabant. Capiatur æquationis Fluxio, & erit $3y^2\dot{y} - ax\dot{y} = 3x^2\dot{x} + ay\dot{x}$, unde $\dot{x} : \dot{y} :: 3y^2 + ax : 3x^2 + ay$, hoc eſt, *HG* ad *GK* vel *DE* : *EH* :: $3y^2 - ax : 3x^2 + ay$, pro y ſubſtitue x ejus valorem, quum eſt x infinite magna, & erit *DE* : *EH* :: $3x^2 - ax : 3x^2 + ax$, & ſumendo rationem ultimam *DE* : *EH* :: $3x^2 : 3x^2$, id eſt, in ratione æqualitatis. Datur igitur Aſymptotos *DH* poſitione. Reſtat jam ut inveniatur in Abſciſsâ punctum *D* per quod tranſit Aſymptotos. Quoniam mox oſtenſum eſt eſſe *DH* : *EH* vel *DE* : $y :: 3y^2 - ax : 3x^2 + ay$, erit $DE = \dfrac{3y^3 - axy}{3x^2 + axy}$

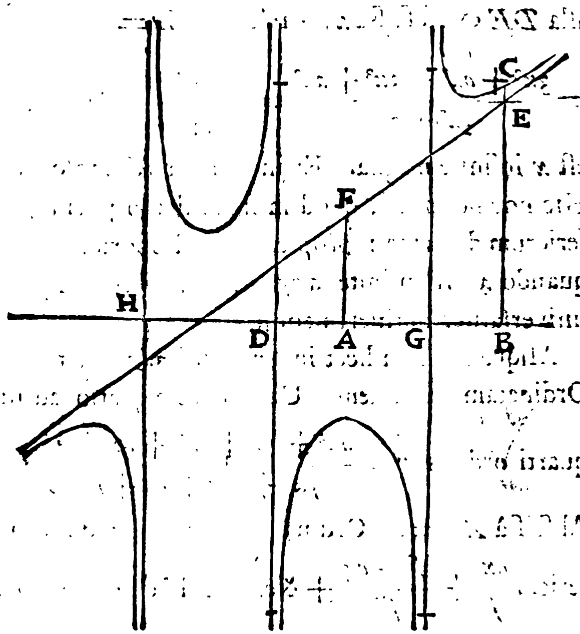
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$= \frac{3x^3 - ax^2}{3y^2 + ax}$ ob y æqualem x. Subducatur jam recta illa DE ex Abſciſsâ x, & reſtabit $AD = x - \frac{3x^3 - ax^2}{3x^2 + ax}$ $= \frac{3x^3 + ax^2 - 3x^3 + ax^2}{3x^2 + ax} = \frac{2ax^2}{3x^2 + ax} = \frac{2}{3}a$; quoniam eſt x infinite magna. Et inde datur Aſymptotos DH. Hic notandum eſt, quod in hoc calculo præſupponitur ſerierum doctrina: propterea quod oportet invenire y quando x eſt infinite magna: hoc vero abſque ſerie univerſaliter obtineri nequit.

Aliquando non licet invenire Aſymptotos reducendo Ordinatam in ſeriem. Ut ſi eſſet æquatio ad lineam quarti ordinis $y = \frac{ax^4 + bx^3 + cx^2 + dx + e}{fx^3 + gx^2 + hx + k}$; ubi eſt Abſciſsâ $AB = x$, Ordinata $BC = y$. Reducatur y in ſeriem $\frac{ax}{f} + \frac{bf - ag}{f^2} +$ &c. Ex hâc ſerie invenietur unica tantum Aſymptotos FE; reliquæ enim Aſymptoti, Ordinatæ parallelæ ex illâ non prodeunt: prodibunt tamen reducendo valorem Abſciſſæ in ſeriem ex dignitatibus Ordinatæ deſcendentibus confectam; ſed facilius hoc modo. Patet enim Ordinatam evadere infinite magnam, adeoque curvæ Aſymptoton, quotieſcunque quantitas $fx^3 + gx^2 + hx + k$ evadit nihil: & hoc ter accidere poteſt, propter Æquationis Cubicæ $fx^3 + gx^2 + hx + k = 0$ tres radices. Nam ſi Æquationis illius radices omnes ſint reales & inæquales, ex

ſunto,

ſunto *AG*, *AD*, *AH* & tres Ordinatæ per puncta *G*, *D*, *H* tranſeuntes erunt totidem Aſymptoti. Quare



in illo caſu curva habet omnino quatuor Aſymptotos & octo crura adjacentia; uti in Schemate videre eſt. At ſi Æquationis illius duæ radices æquales ſint, duæ Aſymptoti coibunt; atque evaneſcunt crura quæ prius inter eas contenta erunt. Et curva habebit duas tantum Aſymptotos cum ſex cruribus. Si æquationis ſupradictæ radices omnes ſint æquales, aut earum duæ imaginariæ, vel coibunt tres Aſymptoti vel duæ evaneſcent, at in utroque caſu figura habebit duas Aſymptotos

ptotos ſeſe ſecantes; in quarum angulis oppoſitis jacent Hyperbolæ oppoſitæ adinſtar Hyperbolæ Conicæ.

In Aſymptotôn rectarum inventione, non ſemper neceſſe habemus ad ſeries recurrere. Nam aſſumi poteſt æquatio univerſalis deſignans omnes curvas generis alicujus, & inde ſerierum ope conſtrui poteſt Canon generalis qui ſufficiat ad inventionem Aſymptotôn Linearum omnium illius ordinis. Ut ſi eſſet æquatio $Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0$ ad Coniſectiones, ubi eſt x Abſciſſa, y Ordinata; Suppono y æqualem huic ſeriei $ax + b + cx^{-1} + \&c.$ Et determinando Coefficientes, uti jam oſtenſum eſt, invenietur eſſe a radix æquationis $Aa^2 + Ba + C = 0$, unde dabitur a; eritque $b = -\frac{Da+E}{2aA+B}$, $c = -\frac{Ab^2+Db+F}{2aA+B}$.

Si æquatio ſit $Ay^3 + Bxy^2 + Cx^2y + Dx^3 + Ey^2 + Fxy + Gx^2 + Hy + Kx + L = 0$ ad Lineas tertii ordinis, erit a radix æquationis: $Aa^3 + Ba^2 + Ca + D = 0$,

$$b = -\frac{Aa^2 + Ba + C}{3Ea^2 + 2Fa + C}$$ atque erit erit $c =$

$$-\frac{3Aab^2 + Bb^2 + Eab + Fb + Ha + K}{3Aa^2 + 2Ba + C}.$$ Quibus expreſſionibus ſemel inventis, invenire licet Aſymptotos rectas harum curvarum abſque recurſione ad ſeries. Et notandum eſt quod a ſemper dat inclinationem Aſymptoti ad Abſciſſam, b dat diſtantiam inter principium Abſciſſæ & punctum in quo Aſymptotos eandem ſecat, c denique oſtendit ad quas Aſymptoton partes jacent

earum,

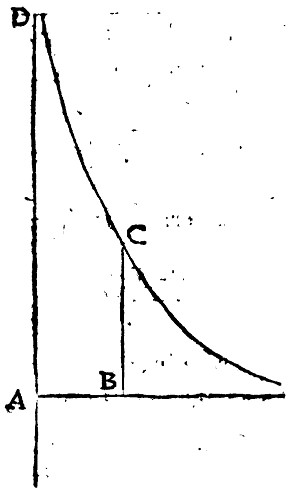
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carum crura. Hæc omnia ex Propoſitione & ejus Corollariis admodum manifeſta ſunt.

PROP. VII. PROBL.

Invenire Numerum & Plagam crurum Aſymptoton aliquam adjacentium.

CAS. 1. Sit primo curva, cujus Aſymptotos recta *AD* per initium Abſciſſæ tranſiens, Abſciſſa $AB = x$, & ipſi *AD* parallela ſit Ordinata $BC = y$. Reducatur y in ſeriem $\frac{A}{x^n} + \frac{B}{x^{n-r}}$ $\frac{C}{x^{n-2r}}$ + &c. eo citius convergentem quo minor eſt x. Et n index ipſius x ſemper erit affirmativus, propterea quod quum eſt x infinite parva, Ordinata y coincidit cum Aſymptoto, & per conſequens eſt infinite magna. Jam ſit x infinite parva, & evadet $y = \frac{A}{x^n}$ accurate, qui valor eſt infinite magnus & affirmativus; quare curva habet crus unum ad eaſdem Aſymptoti *AD* partes jacens cum Ordinata *BC*, & in eaſdem cum Ordinatâ partes ex-



extenſum. Mutari jam concipe ſignum ipſius *x*, & eam etiamnum manere infinite parvam; atque ſi *n* ſit numerus integer vel fractus cum Denominatore impare, figura habebit aliud crus ad alias Aſymptoti partes jacens, & in eaſdem cum priori plagas extenſum, ſi *n* eſt numerus par, at in oppoſitas ſi ſit *n* impar vel fractus cum Denominatore impare. Si vero ſit *n* fractio cum Denominatore pare, ſignum ipſius *x* mutari nequit, at Aſymptotos habebit duo crura ad eaſdem ejus partes jacentia & in plagas oppoſitas ſerpentia.

Caſ. 2. Si Aſymptotos ſit curva, numerus crurum dignoſcitur ex numero valorum Ordinatæ coeuntium, cum abſciſſa eſt infinite magna.

Si alii ſint ſerierum caſus alii itidem erunt crurum caſus, at eorum plagæ & numerus ſemper innoteſcent. Q. E. I.

Coroll. 1. Si *n* ſit numerus integer & par Aſymptotos habet duo crura ad diverſas ejus partes jacentia & in plagas eaſdem progredientia.

Coroll. 2. Si *n* ſit integer & impar, vel fractus cum impare Denominatore, Aſymptotos habet duo crura ad diverſas ejus partes jacentia & in plagas oppoſitas protenſa.

Coroll. 3. Si *n* ſit fractus cum Denominatore pare Aſymptotos habet duo crura ad eaſdem ejus partes jacentia & in plagas oppoſitas ſerpentia.

Coroll. 4. Hinc etiam comparantur longitudines Aſymptotôn inter ſe. Unde conſtabit quaſdam ad ſe invicem datam habere rationem,

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alias vero eſſe aliis infinite majores vel minores.

Coroll. 5. Quamvis omnia crura cum Aſymptotis ſuis tandem coincidere cenſenda ſunt: tamen indiſtantiis æqualibus utcunque magnis aliqua crura ad Aſymptotos ſuas aliis infinities propinquiora accedunt.

PROP. VIII. THEOR.

ÆQuatio $y = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 +$ &c. deſignat figuram habentem duo tantum crura infinita ad eaſdem vel oppoſitas plagas progredientia prout index dignitatis x in termino altiſſimo eſt numerus par vel impar.

Senſus Propoſitionis eſt, quod æquationes $y = a + bx$, $y = a + bx + cx^2 + dx^3$, $y = a + bx + cx^2 + dx^3 + ex^4 + fx^5$, &c. ubi indices terminorum altiſſimorum ſunt numeri impares, deſignant figuras habentes duo tantum crura infinita ad oppoſitas plagas protenſa; & quod æquationes $y = a + bx + cx^2$, $y = a + bx + cx^2 + dx^3 + ex^4$, $y = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6$, &c. ubi terminorum altiſſimorum indices ſunt numeri pares, deſignant Figuras habentes duo tantum crura infinita ad eandem plagam pergentia.

Propoſitio vero ſic demonſtratur.

Concipe Abſciſſam x perpetuo augeri, & ſimul augebitur Ordinata y; ſit x tandem infinite magna, & erit etiam y infinite magna; adeoque

que figura habet crus infinitum. Cum vero Ordinata y eſt infinite magna, pendet ejus ſignum, hoc eſt, plaga cruris infiniti, ex ſigno termini altiſſimi, quoniam in illo caſu is eſt reliquis infinite major. Evadat jam x negativa, id eſt ſumatur Abſciſſa ad alteras partes, & ad illas partes infinitum augeatur; atque etiam augebitur y inin finitum; unde curva habet crus aliud infinitum. Si index termini altiſſimi ſit par, ejus ſignum in utroque caſu idem eſt, ſin impar in uno caſu erit affirmativus, in altero negativus. Cum igitur plaga crurum pendet ex ſigno termini altiſſimi, in primo caſu pergent crura ad plagas eaſdem, in ſecundo ad oppoſitas. Q. E. D.

Pendet itaque tota hujus demonſtationis vis ex termino altiſſimo; adeoque nihil refert quomodo ſeſe habeant termini intermedii. Eodem res redit,utrum affirmentur vel negentur; utrum ſint in æquatione vel non; nam demonſtratio ab illis minime turbabitur.

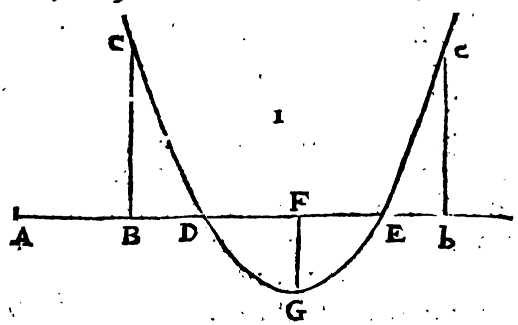
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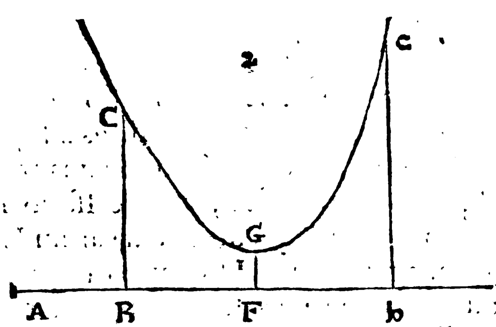
PRopoſitionem hanc eo conſilio præmiſi, ut per eam facilior pateret aditus ad quaſdam æquationum affectiones à nemine quantum ſcio hucuſque ſatis expoſitas, at neceſſario intelligendas ab eo qui ſequentem curvarum Enumerationem aggreditur. Uſus vero Propoſitionis Exemplis ſequentibus ſtatim apparebit in dignoſcendis realibus & imaginariis radicibus æquationum, idque vel calculo, vel deſcribendo cur-

vam; ſed & deſcribendo curvas conſtruuntur æquationes; ſatis quidem expedite, modo quis hâc methodo procedere aſſuetus ſit. Nam ſi æquatio ſit Cubica, ſufficicit invenire quinque vel ſex curvæ puncta; ſi Biquadratica, ſufficit invenire octo vel decem.

Exemplum Primum.

SIT $y = xx + Ax + B$ æquatio deſignans curvam habentem duo crura ad eaſdem plagas protenſa, cujus Abſciſſa $AB = x$, Ordinata





$BC = y$. AB vel ſecat curvam in duobus punctis (*D*, *E Fig.* 1.) vel in nullis (*Fig.* 2.) Sit jam

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jam $y = 0$, vel $xx + Ax + B = 0$, & æquationis illius radices erunt *AD* & *AE* in *Figura* 1*ma*, at in *Figura* 2*da* ſunt imaginariæ. Vides igitur quod, in caſu primo, exiſtente Abſciſsâ minore minimâ radice *AD* vel majore maximâ *AE*, Ordinata correſpodens erit affirmativa: atque Ordinata inter puncta *D* & *E* erecta eſt negativa. Hæc autem omnia vera ſunt ex hypotheſi quod terminus altiſſimus *xx* ſit affirmativus. Igitur *D*, *E* ſunt limites in quibus Ordinata *y* nec affirmatur neque negatur ſed nulla eſt, hoc eſt in quibus quantitas $xx + Ax + B$ nec affirmatur neque negatur, ſed nihil eſt. Et Ordinatæ punctis *D*, *E* proxime & ad diverſas eorum partes jacentes, ſigna contraria ſemper habebunt. In caſu ſecundo quando Abſciſſa minime ſecat curvam, Ordinatæ omnes ejuſdem ſunt ſigni tendentes ad plagam crurum.

Supponamus jam eſſe aliam curvam eandem Abſciſſam *AB* habentem: Ordinatam vero quæ realis ſit quum Ordinata *y* eſt affirmativa, quæque imaginaria ſit quum *y* eſt negativa. Et in caſu primo quando æquationis $x^2 + Ax + B = 0$ radices ſunt reales (*Fig.* 1.) Ordinatæ novæ curvæ per puncta *D* & *E* tranſeuntes ſemper tangent curvam & puncta contactus erunt limites per quos Ordinatæ motu parallelo latæ tranſeunt ipſo temporis momento, quo evadunt poſſibiles ex impoſſibilibus, aut è contra. Atque Ordinatæ inter puncta *D*, *E* erectæ reales aut imaginariæ erunt prout negatur aut affirmatur terminus *xx*. Ordinatæ ad alia quævis Abſciſſæ puncta erectæ

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reales aut imaginariæ erunt prout affirmatur aut negatur terminus ille xx.

In caſu ſecundo ubi Abſciſſa non ſecat curvam, Ordinatæ omnes hujus novæ curvæ reales quidem erunt quando affirmatur xx, ſed omnes prorſus imaginariæ quando negatur idem terminus.

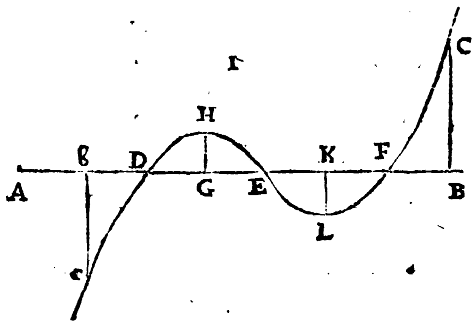
Sit FG ordinata maxima (*Fig.* 1.) inter puncta D, E erecta, at in Fig. 2. omnium minima, & in illo caſu erit $\dot{y}=0$, vel $2xx+Ax=0$, nude $x=-\frac{1}{2}A=AF$: quem valorem in æquatione pro x ſubſtitue, & invenies $y=B-\frac{1}{4}AA=FG$. Et patet quod Abſciſſa ſecat vel non ſecat curvam prout Ordinata illa FG tendit ad contrarias vel eaſdem plagas cum cruribus; hoc eſt, æquationis $xx+Ax+B=0$ radices ſunt poſſibiles quando affirmatur quantitas $\frac{1}{4}AA-B$, & impoſſibiles quando negatur eadem.

Exemplum Secundum.

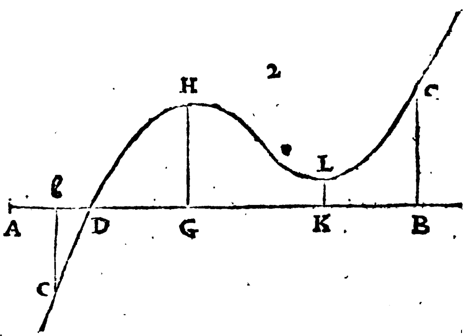
SIT æquatio $y=x^3+Ax^2+Bx+C$ deſignans curvam habentem duo crura in plagas oppoſitas protenſa, cujus Abſciſſa $AB=x$, Ordinata $BC=y$. AB vel ſecat curvam in tribus punctis (*Fig.* 1.) vel in unico (*Fig.* 2, 3.) Sit $(y=)\ x^3+Ax^2+Bx+C=0$, & hujus æquationis radices erunt AD, AE & AF. Igitur ſi radices omnes ſint reales ut in Fig. 1. exiſtente Abſciſsâ minore radice mediâ AE & majore minimâ AD, vel majore maximâ AF, ordinatæ valor reſpectivus erit affirmativus; & ordinatæ ad alia quæ-

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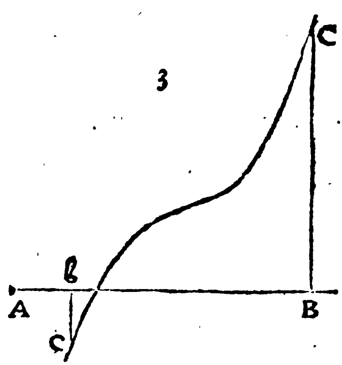
1



2



3



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quævis puncta erecta erunt negativæ. Negativæ enim & affirmativæ evadunt Ordinatæ ſemper per vices. Jam ſit alia curva cujus Ordinata poſſibilis ſit, quando y eſt affirmativa, impoſſibilis vero quando negativa; Et Ordinata illa nova ſubibit vices omnes poſſibilitatis, quas ſubit y mutationis ſignorum $+$ & $-$: ſcilicet tres Ordinatæ ad puncta *D*, *E*, *F* erectæ tangent curvam & puncta contactus erunt limites in quibus y incipit vel deſinit eſſe. Etſi Ordinatæ affirmativæ tendant ad eandem plagam cum Ordinata *BC*, intra limites *D*; *E* continebitur curva partem figuræ conſtituens: jacebit vero altera pars figuræ ad eaſdem partes puncti *F* cum *BC*: & ſemper progredietur in infinitum ab *F* verſus *B*, quoniam unicus illi tantum adeſt limes. In caſu altero, ubi Abſciſſa ſecat curvam in unico puncto (*Fig.* 2. 3.) Si x^3 terminus altiſſimus affirmatur, ordinatæ omnis negativæ aut affirmativæ erunt, prout jacent ad partes puncti *D* eaſdem vel contrarias iis ad quas jacet *A*. Et omnia quæ de primo caſu dicta ſunt, ad hunc caſum mutatis mutandis facile accomodantur.

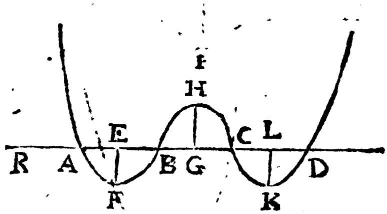
Methodus determinandi Radices reales & imaginarias Æquationes Cubicæ.

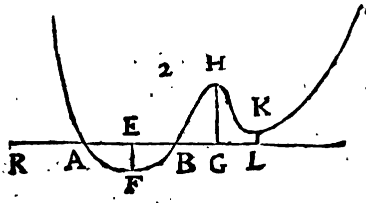
PONE $y = 0$, & erit $3x^2 + 2Ax + B = 0$; cujus æquationis radices reales in *Fig.* 1ma & 2da, ſcilicet *AG*, *AK*, at impoſſibiles in *Fig.* 3^tia^. Igitur ſi æquationes $3x^2 + 2Ax + B = 0$ radices ſint

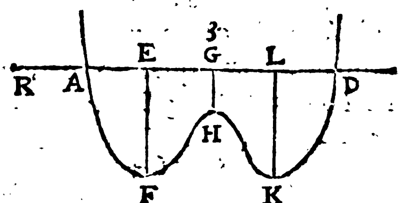
ſint impoſſibiles, id eſt, (per *Exemp.* 1.) Si $\frac{1}{9}AA - \frac{1}{3}B$ vel $\frac{1}{3}AA - B$ ſit negativa quantitas, æquationis $x^3 + Ax^2 + Bx + C$ radices duæ erunt imaginariæ. At ſi æquationis $3x^2 + 2Ax + B$ radices ſint reales ut in *Fig.* 1, 2. æquationis $x^3 + Ax^2 + Bx + C$ radices omnes ſunt reales ubi Ordinatæ GH, KL contraria habent ſigna, *Fig.* 1. & imaginariæ quando eadem habet *Fig.* 2. Pone $DD = AA - 3B$, invenies $GH = C + \frac{2A^3 - 9AB + 2D}{27}$, $LK = C + \frac{2A^3 - 9AB - 2D^3}{27}$. Et hiſce valoribus ſemel inventis, facile eſt invenire quæ radices reales ſunt, quæ non.

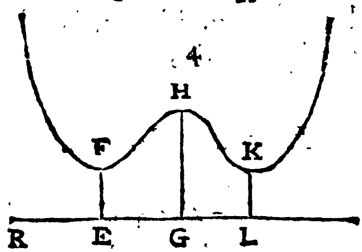
Exemplum Tertium.

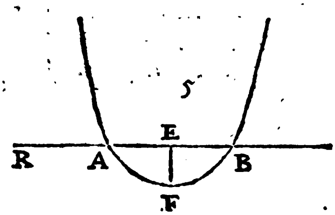
SIT jam $y = x^4 + Bx^3 + Cx^2 Dx + E$ æquatio deſignans Lineam habentem duo crura infinita ad eaſdem plagas, protenſa. Pone $x^4 + Bx^3 + Cx^2 + Dx + E = 0$; hujus æquationis radices quatuor (*Fig.* 1.) ſunt RA, RB, RC, RD,

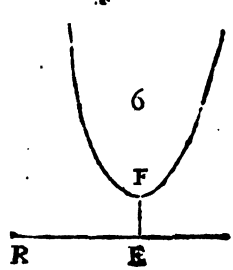



2
H
K
E
R A B G L
F


3
E G L
R A D
H
F K


4
H
F K
R E G L


5
E
R A B
F


6
F
R E

at in *Fig.* 2, 3, 5, duæ radices impoſſibiles, & in *Fig.* 4, 6. omnes ſunt impoſſibiles. Pone $y=0$, & erit $4x^3+3Bx^2+2Cx+D=0$, cujus æquationis tres radices (*Fig.* 1, 2, 3, 4.) ſunt *RE*, *RG*, *KL*: at in *Fig.* 5, 6. æquatio illa unicam tantum habet radicem *RE*. In primo caſu (*Fig.* 1) ſi Ordinatarum *EF*, *GH*, *LK*, duæ ſunt negativæ, & tertia affirmativa, æquatio $x^4+Bx^3+Cx^2+Dx\,E=0$ habet quatuor radices poſſibiles. Si ordinatarum duæ ſint affirmativæ & tertia negativa (*Fig.* 2.) vel ſi omnes tres ſint negativæ (*Fig.* 3.) æquatio non habet niſi duas radices reales: Si ordinatæ omnes ſint affirmativæ (*Fig.* 4.) radices omnes erunt impoſſibiles.

In ſecundo caſu quando æquationis $4x^3+3Bx^2+2Cx+D=0$ radix unica erat poſſibilis (*Fig.* 5, 6.) ſi ordinata *EF* ſit negativa (*Fig.* 5.) radices duæ erunt reales tantum: at ſi ordinata illa ſit affirmativa (*Fig.* 6.) omnes radices erunt imaginariæ. Hi ſunt Caſus Æquationis Biquadraticæ.

Unde inveniri poteſt Canon generalis pro dignoſcendis radicibus realibus & imaginariis æquationum Biquadraticarum, ut in *Exemplis primo* & *ſecundo* fecimus pro æquationibus Quadraticis & Cubicis. Hoc vero prolixum admodum requirit calculum. Sufficiat quod novimus quomodo tractanda eſt æquatio particularis, ubi calculus non erit adeo laborioſus. Eadem methodus ad omnes æquationes extenditur. Si ſit nova curva ordinatam habens realem quando

hujus eſt affirmativa & imaginariam quando hujus curvæ ordinata eſt negativa; vices omnes poſſibilitates & impſſibilitatis facillime innoteſcunt per ea quam diximus in *Exemplis duobus primis.*

Ex hiſce Exemplis ſatis apparet numerum radicum impoſſibilium ſemper eſſe parem. Item ſequitur, quod ſi ſit æquatio $x^n + Bx^{n-1} + Cx^{n-2} + \&c. = 0$. & ſit $B^2 - C \times n^2 \times \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times \&c.$ quantitas negativa, æquatio illa ad minimum habebit duas radices imaginarias. Continuanda vero eſt ſeries $n^2 \times \overline{n-2} \times \overline{n-3} \times \&c.$ donec n per continuam unitatis ſubductionem tandem exhauriatur. Et ſi deſit terminus ſecundus Bx^{n-1}, & Cx^{n-2} huic proximus ſit affirmativus, æquatio ad minimum habebit duas radices imaginarias.

Ex Propoſitionibus hactenus traditis invenire licet genus, poſitionem, plagam & numerum crurum infinitorum: & ex hâc Propoſitione ejuſque Exemplis invenies quæ crura conjunguntur: adeoque invenies formam curvæ. Et faciendo rectam gyrare circa punctum aliquod idoneum & ſecare Parabolas quas in hâc Propoſitione deſcripſimus, habebuntur omnes caſus alicujus æquationis radicum poſſibiles; hoc eſt, omnes formæ curvarum quas æquatio generalis deſignat: & inde enumerantur Linearum ſpecies, ſiquidem ſpecies curvarum non tam ab ipſarum intimâ naturâ, quam à forma pendere volumus.

Pro-

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Propoſitiones quæ ſequuntur continent conſimiles aliquot curvarum Rationalium proprietates ab æquationum naturâ immediate fluentes.

PROP. IX. PROBL.

Invenire numerum punctorum quæ determinant Lineam alicujus ordinis.

Linea quævis deſcribi poteſt per tot puncta, quot ſunt Coefficientes in æquatione generaliſſimâ eam definiente & non plura; ut conſtat ex methodo D. *Newtoni* univerſali deſcribendi curvas per data totidem puncta quot eas determinant: cujus ſpecimen dedit in ſectionibus Conicis ad Algebræ ſuæ Problemata 54, 57. Et hactenus annotatum eſt, quod exiſtente *n* numero dimenſionum curvæ, erit $\frac{n^2+3n}{2}$ numerus Coefficientium in æquatione generaliſſimâ Lineas omnes alicujus ordinis definiente; & proinde etiam erit $\frac{n^2+3n}{2}$ numerus punctorum determinantium curvam cujus dimenſio eſt *n*.

Itaque recta determinatur ex duobus punctis, ſectio Coni ex quinque; Linea tertii Ordinis ex novem, Linea quarti ex quartuordecem. Et ſic porro.

Coroll. 1. Ex dato numero punctorum determinantium dabitur dimenſio curvæ. Sit enim

m pun-

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m punctorum numerus, & erit $m = \frac{n^2 + 3n}{2}$.

Unde & viciſſim invenies $n = \frac{-3 + \sqrt{8m + 9}}{2}$.

Ut ſi eſſet $m = 20$, invenies $n = 5$.

Coroll. 2. Ex hâc Propoſitione invenies numerum punctorum quæ determinant Lineam aliquam particularem, ſi poſſibile ſit, ſunt enim quædam Lineæ quas nullus punctorum numerus determinat.

Exempla.

1. SIT $y = ax + b \pm \sqrt{cx + d}$ æquatio generaliſſima ad Parabolam Conicam: quum igitur quatuor tantum ſunt Coefficientes, Parabola determinatur ex quatuor punctis. Quatuor puncta non ſufficiunt ad determinandam Hyperbolam aut Ellipſin ſed quinque nimia ſunt.

2. Item ſectio Coni cujus datur Diameter, & angulus Ordinatarum, determinatur ex tribus punctis, Parabola ex duobus; duo non ſufficiunt ad determinandam Hyperbolam aut Ellipſin, ſed tria nimia ſunt.

3. Hyperbola Conica, datâ poſitione ejus Aſymptoto, determinatur ex quatuor punctis.

4. Parabola Conica, datâ poſitione ejus axe, determinatur ex tribus punctis.

5. Para-

5. Parabolæ quinque *divergentes*, datâ plagâ crurum, determinantur ex ſex punctis; Parabola *Carteſii* ex quinque; Parabola *Cubica* ex quatuor.

PROP. X. THEOR.

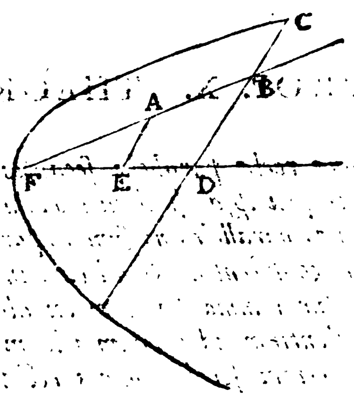
Ducantur duæ rectæ parallelæ ſecantes curvam in tot punctis quot ipſa eſt dimenſionum; recta quæ ita ſecat has parallelas ut ſumma partium ex uno ejus latere conſiſtentium & ad curvam terminatarum æquetur ſummæ partium ex altero ejus latere conſiſtentium ad curvam itidem terminatarum, ita etiam ſecabit omnes rectas hiſce parallelas.

SIT enim curvæ alicujus Abſciſſa $AB=x$, Ordinata $BC=y$, Æquationis relationem inter x & y definientis terminus ſecundus ſit $\overline{ax+b}\times y^{n-1}$: ſume in rectâ AB, $AF=\frac{-b}{a}$, & ordinatæ parallelam $AE=\frac{-b}{n}$; junge EF; ſi ea ſumatur pro Abſciſsâ, dico ſummam Ordinatarum ex unâ ejus parte æquari ſummæ Ordinatarum ex alterâ parte.

Nam ſit $y+\overline{ax+b}\times y^{n-1}+$ &c. $=0$ æquatio exprimens relationem inte x, y. Producatur CB ſecans FE in D & ſit abſciſſa nova $ED=z$,

Or-

Ordinata nova $DC = v$: ponatur $AB :: ED :: A : 1$, id eſt $x : z :: A : 1$, unde $x = Az$. $FA : AE$



$:: FB : BD$; vel $\frac{-b}{a} : \frac{-b}{n} :: Az + \frac{b}{a} : BD$ $= \frac{aAz+b}{n}$. $BC = DC - DB = v - \frac{aAz+b}{n} = y$, unde (per Theor. D. *Neutoni*) $y^n = v^n - \overline{aAz+b} \times v^{n-1}$ &c. $y^{n-1} = v^{n-1}$ &c. Hoſce valores ſubſtitue in æquatione $y^n + \overline{ax+b} \times y^{n-1} +$ &c. $= 0$, & videbis v^{n-1} evaneſcere, id eſt æquationis terminum ſecundum deeſſe: igitur valores ipſius x erunt partim negativi & partim affirmativi, & ſumma affirmativorum æquabitur ſummæ negativorum: vel, quod perinde eſt, æquantur ſummæ Ordinatarum ex Abſciſſa ad curvam in eaſdem partes extenſarum. Q. E. D.

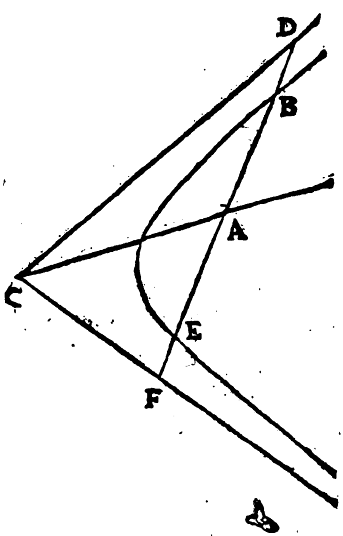
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Recta quæ ita ſecat ordinatas appelletur Curvæ *Diameter*.

Coroll. 1. Duc rectas duas parallelas ſecantes Lineam ſecundi ordinis in duobus punctis, recta quæ has biſecat, biſecabit omnes illis parallelas. Adeoque æquatio $yy = ax^2 + bx + c$ deſignat omnes Lineas ſecundi ordinis.

Coroll. 2. Duc rectas duas quaſvis parallelas, ſecantes Lineam tertii ordinis in tribus punctis; recta quæ ita ſecat has parallelas, ut ſumma duarum partium ex uno ejus latere conſiſtentium, & ad curvam terminatarum, æquetur parti tertiæ ex altero ejus latere conſiſtenti, & ad curvam terminatæ, ita etiam ſecabit omnes rectas hiſce parallelas.

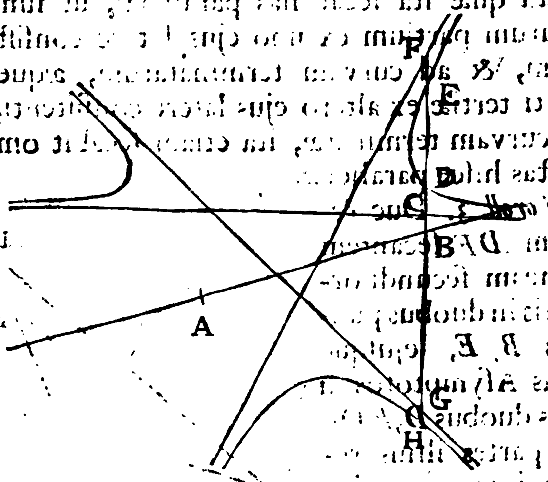
Coroll. 3. Duc rectam *DF* ſecantem Lineam ſecundi ordinis in duobus punctis *B*, *E*, ejuſque duas Aſymptotos in aliis duobus *D*,*F*, Dico partes illius rectæ interceptas inter Aſymptotos & earum crura ſibi invicem æquari. Duc Diametrum *CA* biſecantem *BE* omneſque rectas illi parallelas. Quia curva



coincidit cum Aſymptotis ad diſtantiam infini-

tam, coeunt in illo caſu puncta *B*, *D*; *E*, *F*: adeoque Diameter biſecat ordinatas in Aſymptotis terminatas in diſtantiâ infinitâ: unde per naturam rectæ in omni diſtantiâ eas biſecabit, hoc eſt, ubique *AD* = *AF*, ſed *AB* = *AE*, ergo *BD* = *EF*. Q. E. D.

Coroll. 4. Duc rectam *FH* ſecantem Lineam tertii ordinis in tribus punctis *E*, *D*, *H*; ut & tres ejus Aſymptotos in tribus aliis *F*, *C*, *G*.



Dico *FE*, *GH* partes duas hujus rectæ inter Aſymptotos & crura interceptas & ab Aſymptotis ad curvam in eaſdem plagas extenſas æquari parti tertiæ *CD* inter Aſymtoton tertiam & ejus crus interceptæ, & ab Aſymptoto ad curvam in oppoſitas plagas extenſæ. Ducatur enim Diameter *AB* quæ ita ſecat ordinatas, ut ſit

ſit $BD+BE=BH$; quoniam in diſtantiâ infinitâ Aſymptoti coincidunt cum ſuis cruribus, coeuntibus punctis *F*, *E*; *C*, *D*; *G*, *H*; Diameter *AB* ita etiam ſecabit ordinatas in Aſymptotis terminatas in diſtantiâ infinitâ ut ſit $BC+BF=BG$: at ſi hoc accidat in quâlibet diſtantiâ, accidet in omni, ergo eſt univerſaliter $BC+BF=BG$, ſed $BD+BE=BH$; ergo demendo æqualia ab æqualibus, erit $BD-BC+BE-BF=BH-BG$, hoc eſt, $CD+EF=GH$, vel $CD=EF+GH$. Q. E. D.

Eâdem facilitate ſimile demonſtratur de curvis ſuperiorum ordinum.

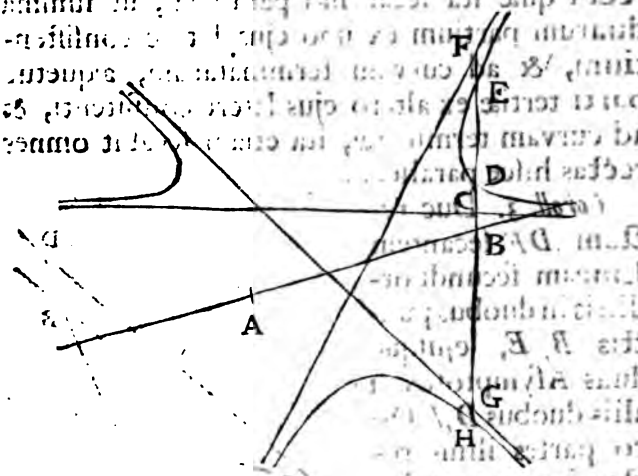
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HAnc Propoſitionem demonſtravi conſiderando coefficientem termini ſecundi eſſe ſummam omnium radicum ſub ſignis propriis collectam. Coefficiens tertii termini eſt factum ſub ſingulis duabus radicibus, coefficiens quarti ſub ſingulis tribus, quinti ſub ſingulis quatuor, & ſic in infinitum. Unde faciliime ſequitur ſeries Theorematum ſequentium ad libitum continuanda.

1. Ducantur quinque parallelæ ſecantes Lineam tertii vel ſuperioris ordinis in tot punctis quot curva habet dimenſiones: Sectio Conica vel linea recta, quæ ita ſecat has parallelas, ut ſumma rectangulorum ſub partibus earum inter curva ſectionem Conicam vel rectam interceptis, & à curvâ ad ſectionem Coni vel rectam in

tam, coeunt in illo caſu puncta *B*, *D*; *E*, *F*: adeoque Diameter biſecat ordinatas in Aſymptotis terminatas in diſtantiâ infinitâ: unde per naturam rectæ in omni diſtantiâ eas biſecabit, hoc eſt, ubique *AD* = *AF*, ſed *AB* = *AE*, ergo *BD* = *EF*. Q. E. D.

Coroll. 4. Duc rectam *FH* ſecantem Lineam tertii ordinis in tribus punctis *E*, *D*, *H*; ut & tres ejus Aſymptotos in tribus aliis *F*, *C*, *G*;



Dico *FE*, *GH* partes duas hujus rectæ inter Aſymptotos & crura interceptas & ab Aſymptotis ad curvam in eaſdem plagas extenſas æquari parti tertiæ *CD* inter Aſymtoton tertiam & ejus crus interceptæ, & ab Aſymptoto ad curvam in oppoſitas plagas extenſæ. Ducatur enim Diameter *AB* quæ ita ſecat ordinatas, ut

fit

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ſit $BD + BE = BH$; quoniam in diſtantiâ infinitâ Aſymptoti coincidunt cum ſuis cruribus, coeuntibus punctis F, E; C, D; G, H; Diameter AB ita etiam ſecabit ordinatas in Aſymptotis terminatas in diſtantiâ infinitâ ut ſit $BC + BF = BG$: at ſi hoc accidat in quâlibet diſtantiâ, accidet in omni, ergo eſt univerſaliter $BC + BF = BG$, ſed $BD + BE = BH$; ergo demendo æqualia ab æqualibus, erit $BD - BC + BE - BF = BH - BG$, hoc eſt, $CD - EF = GH$, vel $CD = EF + GH$. Q. E. D.

Eâdem facilitate ſimile demonſtratur de curvis ſuperiorum ordinum.

Scholion.

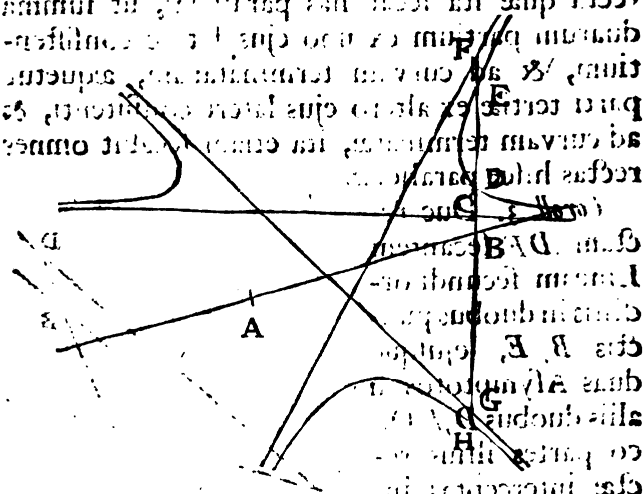
HAnc Propoſitionem demonſtravi conſiderando coefficientem termini ſecundi eſſe ſummam omnium radicum ſub ſignis propriis collectam. Coefficiens tertii termini eſt factum ſub ſingulis duabus radicibus, coefficiens quarti ſub ſingulis tribus, quinti ſub ſingulis quatuor, & ſic in infinitum. Unde faciliime ſequitur ſeries Theorematum ſequentium ad libitum continuanda.

1. Ducantur quinque parallelæ ſecantes Lineam tertii vel ſuperioris ordinis in tot punctis quot curva habet dimenſiones: Sectio Conica vel linea recta, quæ ita ſecat has parallelas, ut ſumma rectangulorum ſub partibus earum inter tv[illegible] Conicam vel rectam interce-[illegible] d ſectionem Coni vel rectam

 in

tam, coeunt in illo caſu puncta *B*, *D*; *E*, *F*: adeoque Diameter biſecat ordinatas in Aſymptotis terminatas in diſtantiâ infinitâ: unde per naturam rectæ in omni diſtantiâ eas biſecabit, hoc eſt, ubique $AD = AF$, ſed $AB = AE$, ergo $BD = EF$. Q. E. D.

Coroll. 4. Duc rectam *FH* ſecantem Lineam tertii ordinis in tribus punctis *E*, *D*, *H*; ut & tres ejus Aſymptotos in tribus aliis *F*, *C*, *G*.



Dico *FE*, *GH* partes duas hujus rectæ inter Aſymptotos & crura interceptas & ab Aſymptotis ad curvam in eaſdem plagas extenſas æquari parti tertiæ *CD* inter Aſymtoton tertiam & ejus crus interceptæ, & ab Aſymptoto ad curvam in oppoſitas plagas extenſæ. Ducatur enim Diameter *AB* quæ ita ſecat ordinatas, ut

ſit

ſit $BD+BE=BH$; quoniam in diſtantiâ infinitâ Aſymptoti coincidunt cum ſuis cruribus, coeuntibus punctis F, E; C, D; G, H; Diameter AB ita etiam ſecabit ordinatas in Aſymptotis terminatas in diſtantiâ infinitâ ut ſit $BC+BF=BG$: at ſi hoc accidat in quâlibet diſtantiâ, accidet in omni, ergo eſt univerſaliter $BC+BF=BG$, ſed $BD+BE=BH$; ergo demendo æqualia ab æqualibus, erit $BD-BC+BE-BF=BH-BG$, hoc eſt, $CD-EF=GH$, vel $CD=EF+GH$. Q. E. D.

Eâdem facilitate ſimile demonſtratur de curvis ſuperiorum ordinum.

Scholion.

HAnc Propoſitionem demonſtravi conſiderando coefficientem termini ſecundi eſſe ſummam omnium radicum ſub ſignis propriis collectam. Coefficiens tertii termini eſt factum ſub ſingulis duabus radicibus, coefficiens quarti ſub ſingulis tribus, quinti ſub ſingulis quatuor, & ſic in infinitum. Unde faciliime ſequitur ſeries Theorematum ſequentium ad libitum continuanda.

1. Ducantur quinque parallelæ ſecantes Lineam tertii vel ſuperioris ordinis in tot punctis quot curva habet dimenſiones: Sectio Conica vel linea recta, quæ ita ſecat has parallelas, ut ſumma rectangulorum ſub partibus earum inter curva ſectionem Conicam vel rectam interceptis, & à curvâ ad ſectionem Coni vel rectam

in eaſdem plagas extenſis, æquetur ſummæ rectangulorum ſub partibus earundem parallelarum ad alteras ſectionis Coni vel rectæ plagas à curvâ extenſis, & in curva terminatis, ita ſecabit omnes rectas hiſce parallelas.

2. Ducantur novem parallelæ ſecantes curvam quarti vel ſuperioris ordinis in tot punctis quot curva eſt dimenſionum. Ducatur Linea tertii vel inferioris ordinis ſecans has novem parallelas: & ſi componantur Parallepipeda ſub ſingulis tribus partibus harum parallelarum inter curvam & Lineam tertii vel ordinis inferioris interceptis: atque in unaquâque novem parallelarum, ſi Parallelepipeda quæ prodeunt affirmata, æqualia deprehendantur iis quæ prodeunt negativa; idem accidet in omnibus rectis prioribus novem parallelis. Et ſic porro.

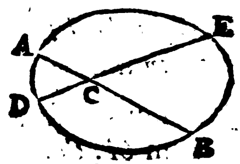
Hæ Lineæ quæ ita ſecant parallelas, tanquam curvarum Diametri (ut ita dicam) quodammodo conſiderari poſſunt.

PROP. XI. THEOR.

Sit AEBD *Linea ſecundi ordinis, quam ſecat recta* AB *in duobus punctis* A, B; *ut & recta* DE *in duobus aliis* D, E; *harum concurſus ſit* C. *Dico eſſe* AC×CB *ad* DC×CE *in ratione datâ; modo detur rectarum* AB, DE *inclinatio ad ſe invicem.*

SUpponamus enim Abſciſſam $AC = x$, & rectarum DC, CE quamlibet ambigue deſignare ordinatam y: atque curva hujuſmodi æquatione

tione $y^2 + ax + bxy + cx^2 - dx = 0$ deſignabitur, in quâ quantitas determinata non reperietur, quoniam principium Abſciſſæ eſt in curva. Ut inveniatur *AB* ſit $y = 0$, & evaneſcent termini in quibus ea reperitur, atque erit $cx^2 - dx = 0$, unde in illo caſu eſt $x = \frac{d}{c} = AB$, ergo $AB - AC = \frac{d}{c} - x = BC$:



cujus ſignum mutetur quoniam puncta *A*, *B* jacent ad partes puncti *C* contrarias, & erit $BC = x - \frac{d}{c}$. Et inde $AC \times BC = xx - \frac{dx}{c}$. Conſtat vero per naturam æquationum, terminum ultimum in quo radix non reperitur, eſſe factum ſub omnibus radicibus; hoc eſt $DC \times CE = cx^2 - dx$, & hoc rectangulum eſt ad $AC \times CB = xx - \frac{dx}{c}$ ut *c* ad unitatem: at ſervatâ rectarum *AB*, *DE* inclinatione ad invicem, non mutabitur quantitas *c*, ergo neque dicta ratio facti $AC \times CB$ ad $DC \times CE$. Q. E. D.

Ex hac propoſitione tanquam totidem Corollaria fluunt omnia, quæ tradere ſolent authores de ſectionum Conicarum Diametris, Verticibus, Lateribus rectis & tranverſis, & ratione contentorum ſub parallelarum ſegmentis.

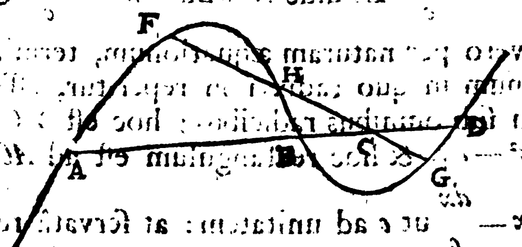
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PROP. XII.

Sit AFG *Linea tertii ordinis, quam ſecet recta* AD *in tribus punctis* A, B, D, *ut &* FG *in tribus aliis* F, G, H: *harum concurſus ſit* C. *Dico eſſe* AC×BC×DC *ad* FC×GC×HC *in ratione datâ; modo detur rectarum* AD, FG *ad ſe invicem inclinatio.*

POnamus eſſe Abſciſſam $AC = x$, & rectarum *CF*, *CH*, *CG* quamlibet ambigue deſignare ordinatam y: & curva hujuſmodi æquatione deſignabitur $y^3 + \overline{ax + b} \times y^2 + \overline{cx^2 + dx + e} \times y = fx^3 - gx^2 + hx$; ubi quantitas data non reperietur propterea quod initium Abſciſſæ eſt in curva. Sit $y = 0$, atque erit $fx^3 - gx^2 + hx = 0$, unde in



illo caſu eſt $x = \frac{g}{2f} \pm \sqrt{\frac{gg}{4ff} - \frac{h}{f}}$, hoc eſt $AB = \frac{g}{2f} - \sqrt{\frac{gg}{4ff} - \frac{h}{f}}$ & $AD = \frac{g}{2f} + \sqrt{\frac{gg}{4ff} - \frac{h}{f}}$; ergo $BC = x - \frac{g}{2f} + \sqrt{\frac{gg}{4ff} - \frac{h}{f}}$ & $CD = x - \frac{g}{2f}$

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$-\sqrt{\frac{gg}{4ff}-\frac{h}{f}}$ cum ſigno mutato. Unde eſt $AC \times BC \times DC = x^3 - \frac{gxx}{f} + \frac{hx}{f}$. Sed per naturam æquationum eſt $FC \times HC \times GC = fx^3 - gx^2 + hx$; ideoque ſolidum prius eſt ad poſterius ut unitas ad f: at ſervata inclinatione rectarum AD, EG, dabitur quantitas f, ergo etiam dicta ratio ſolidi $AC \times BC \times CD$ ad $FG \times GC \times HC$. Q. E. D.

Hinc tam facile conſequuntur ea quæ tradidit D. *Neutonus* de Linearum tertii Ordinis Diametris, Verticibus, Lateribus rectis & tranſverſis, ratione contentarum ſub parallelarum ſegmentis, atque alia plurima; ut eadem plenius oſtendere neceſſarium haud duxerim.

Sufficiat hic obiter annotare, quod hâc methodo univerſali procedendo, ſcilicet argumentando à naturis æquationum, pateſcunt non ſolum ſectionum Coni proprietates, quas tanto labore adinvenerunt veteres, & tot ambagibus demonſtratas dederunt, idque methodo quæ ad alias curvas extendi nequit; ſed & proprietates curvarum omnium ordinum ſuperiorum.

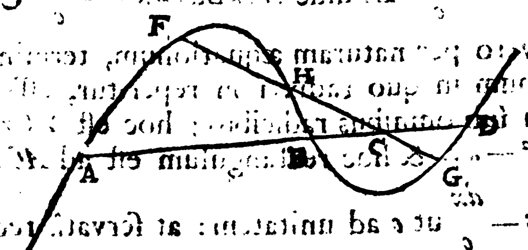
Hiſce præmiſſis, pergerem ad Enumerationem Linearum tertii ordinis; ſed ob rei analogiam enumerare licet eas ſecundi. Hæ (per *Coroll.* 1. *Prop.* 10.) reducuntur omnes ad æquationem $yy = ax^2 + bx + c$; quæ æquatio, ut ſtatim apparebit, deſignat Hyperbolam, Ellipſin aut Parabolam, prout terminus axx affirmativus eſt, negativus, vel nullus.

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PROP. XII.

Sit AFG *Linea tertii ordinis, quam ſecet recta* AD *in tribus punctis* A, B, D; *ut &* FG *in tribus aliis* F, G, H: *harum concurſus ſit* C. *Dico eſſe* AC×BC×DC *ad* FC×GC×HC *in ratione datâ; modo detur rectarum* AD, FG *ad ſe invicem inclinatio.*

POnamus eſſe Abſciſſam $AC = x$, & rectarum CF, CH, CG quamlibet ambigue deſignare ordinatam y: & curva hujuſmodi æquatione deſignabitur $y^3 + \overline{ax+b}\,y^2 + \overline{cx^2+dx+e}\,y = fx^3 - gx^2 + hx$; ubi quantitas data non reperietur propterea quod initium Abſciſſæ eſt in curva. Sit $y = 0$, atque erit $fx^3 - gx^2 + hx = 0$, unde in



illo caſu eſt $x = \frac{g}{2f} \pm \sqrt{\frac{gg}{4ff} - \frac{h}{f}}$: hoc eſt $AB = \frac{g}{2f} - \sqrt{\frac{gg}{4ff} - \frac{h}{f}}$ & $AD = \frac{g}{2f} + \sqrt{\frac{gg}{4ff} - \frac{h}{f}}$; ergo $BC = x - \frac{g}{2f} + \sqrt{\frac{gg}{4ff} - \frac{h}{f}}$ & $CD = x - \frac{g}{2f}$

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$-\sqrt{\frac{gg}{4ff}-\frac{h}{f}}$ cum ſigno mutato. Unde eſt $AC \times BC \times DC = x^3 - \frac{gx^2}{f} + \frac{hx}{f}$. Sed per naturam æquationum eſt $FC \times HC \times GC = fx^3 - gx^2 + hx$; ideoque ſolidum prius eſt ad poſterius ut unitas ad f: at ſervatâ inclinatione rectarum AD, FG, dabitur quantitas f, ergo etiam dicta ratio ſolidi $AC \times BC \times CD$ ad $FG \times GC \times HC$. Q. E. D.

Hinc tam facile conſequntur ea quæ tradidit D. *Neutonus* de Linearum tertii Ordinis Diametris, Verticibus, Lateribus rectis & tranſverſis, ratione contentarum ſub parallelarum ſegmentis, atque alia plurima; ut eadem plenius oſtendere neceſſarium haud duxerim.

Sufficiat hic obiter annotare, quod hâc methodo univerſali procedendo, ſcilicet argumentando à naturis æquationum, pateſcunt non ſolum ſectionum Coni proprietates, quas tanto labore adinvenerunt veteres, & tot ambagibus demonſtratas dederunt, idque methodo quæ ad alias curvas extendi nequit; ſed & proprietates curvarum omnium ordinum ſuperiorum.

Hiſce præmiſſis, pergerem ad Enumerationem Linearum tertii ordinis; ſed ob rei analogiam enumerare licet eas ſecundi. Hæ (per *Coroll.* 1. *Prop.* 10.) reducuntur omnes ad æquationem $yy = ax^2 + bx + c$; quæ æquatio, ut ſtatim apparebit, deſignat Hyperbolam, Ellipſin aut Parabolam, prout terminus axx affirmativus eſt, negativus, vel nullus.

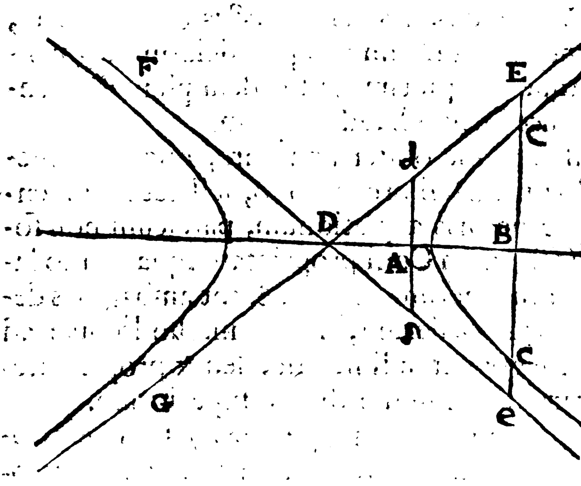
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Enumeratio Linearum ſecundi Ordinis.

PROP. XIII. THEOR.

Æquatio $yy = ax^2 + bx + c$ *deſignat figuram habentem quatuor crura infinita ad duas Aſymptotas rectas jacentia.*

Liquet fere hæc Propoſitio ex *Exemplo primo Scholii Prop.* 8. ſed argumento magis diſtincto ſic evincitur.



Augeatur x in infinitum, & duo valores ordinatæ y hinc inde æquales etiam augebuntur in infinitum; quare figura habet duo crura infinita. Mutetur ſignum ipſius x, hoc eſt ſumatur Abſciſſa ad alteras partes, & æquatio erit hæc $y^2 = ax^2 - bx + c$, ubi patet quod augendo x in

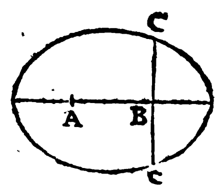
in infinitum, etiam augebitur y in infinitum: nam terminus affirmativus ax^2 erit omnibus reliquis multo major, modo x ſit admodum magna: quare curva habet alia duo crura, & omnino quatuor.

Reducatur y in ſeriem hujuſmodi convergentem $y = \pm x\sqrt{a} \pm \frac{b}{2\sqrt{a}} \pm \frac{4ac - bb}{8ax\sqrt{a}}$ &c. Unde (per *Prop.* 6.) Ordinata Aſymptoti erit $\pm x\sqrt{a} \pm \frac{b}{2\sqrt{a}}$. Igitur pro Aſymptotis habet hæc curva duas rectas ex Abſciſsâ hinc inde æqualiter jacentes. Nam ſume $AD = \frac{-b}{2a}$, in Abſciſsâ; per initium Abſciſſæ duc $dA\delta$ ordinatæ parallelam, in quâ ſint Ad, $A\delta$, æquales $\frac{b}{2\sqrt{a}}$, ad contrarias puncti A partes ſumptæ; junge Dd, $D\delta$, erunt illæ duæ Aſymptoti. Ideoque Figura habet quatuor crura Hyperbolica ad duas rectas jacentia. Q. E. D.

Coroll. 1. Si bb majus ſit $4ac$, crura jacent in angulis EDe, FDG; ſin minus jacent in angulis hiſce deinceps: ut conſtat ex *Coroll.* 2. *Prop.* 6.

Coroll. 2. Hæc Figura (per *Coroll.* 1. *Prop.* 4.) ejus Aſymptotos non decuſſat, adeoque (per *Coroll.* 2. *Prop.* 1.) crura quæ in eodem angulo jacent, ductu continuo ſemper conjunguntur.

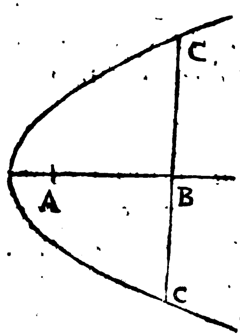
Coroll. 3. Unde hæc Figura ſemper conſtat ex Hyperbolis duabus Inſcriptis, in Aſymptoton angulis oppoſitis, jacentibus; & proinde unicam tantum ſpeciem conſtituit. Quæ eſt ſpecies prima Linearum ſecundi ordinis.



Coroll. 4. Si terminus ax^2 ſit negativus, curva nequit excurrere in infinitum: ergo (per *Prop.* 1.) in ſe redit; & conſtat ex Ovali unicâ. Quæ eſt ſpecies ſecunda. Hoc *Corollarium* facillime colligitur ex *primo Exemplo Scholii Propoſitionis octavæ.*

PROP. XIV. THEOR.

Si deſit terminus ax^2, *æquatio* $y^2 = * bx + c$ *deſignat figuram habentem duo tantum crura infinita Parabolica in eandem plagam extenſa.*



Patet enim quod augendo Abſciſſam x in infinitum, ſimul augebuntur ordinatæ y valores, ergo curva habet duo crura infinita, & in eandem plagam protenſa; quoniam exiſtente x infinite magna, ordinata y eſt eâ infinite minor. Quod ſi mutetur ſignum ipſius x, æquatio evadet $y^2 = -bx + c$; poſtquam ergo eouſque augetur x, ut ſit $bx = c$; cur-

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curva ad illas partes ulterius pergere nequit, quia erit quadratum ordinatæ negativa quantitas, & inde ordinata ipſa impoſſibilis. Igitur curva habet duo tantum crura. Per methodum ſerierum

eſt $y = \pm \sqrt{bx} \pm \frac{c}{2\sqrt{bx}} + \&c.$ Unde (per *Prop.* 6.)

$\sqrt{bx}$ eſt Ordinata Aſymptoti ad Abſciſſam x pertinens, quumque hæc non ſit ordinata rectæ, crura ſunt Parabolica. Q. E. D.

Coroll. Crura hujus curvæ ſunt ſui generis ſimpliciſſima. Hæc figura conſtituit Linearum ſecundi ordinis ſpeciem tertiam; Et patet eas eſſe tres Coni ſectiones.

Enumeratio Linearum tertii Ordinis.

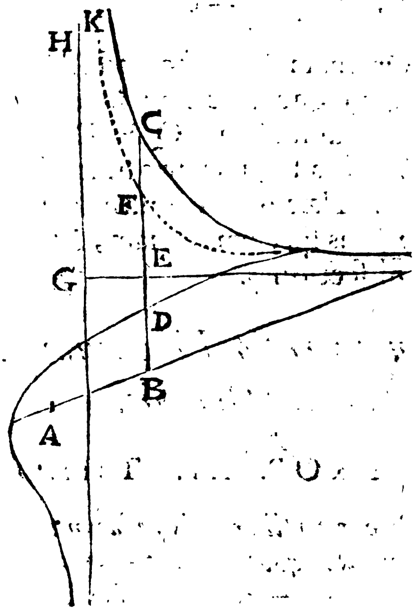
PROP. XV. THEOR.

Omnes Lineæ tertii ordinis reducuntur ad hos æquationum caſus quatuor, $xy^2 - ey = ax^3 + bx^2 + cx + d$, $xy = ax^3 + bx^2 + cx + d$, $y^2 = ax^3 + bx^2 + cx + d$, & $y = ax^3 + bx^2 + cx + d$.

Æquatio $\overline{z + a} \times v^2 = \overline{bz^2 + cz + d} \times v + ez^3 + fz^2 + gz + h$ (per *Coroll.* 1. *Prop.* 5.) comprehendit omnes Lineas *ſecundi* ordinis: unde, ſi probavero illam æquationem ſemper reduci poſſe ad unam quatuor dictarum formarum, conſtabit Propoſitio.

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Caſ. 1. Si æquationis terminus nullus déſit, divide eam per $z+a$ coefficientem ipſius v^2, & erit $v^2 = \frac{\overline{bz^2+cz+d} \times v + ez^3 + fz^2 + gz + h}{z+a}$, atque ex-



trahendo radicem $v = \frac{bz^2+cz+d}{2z+2a} \pm p$, ubi p eſt latus quadratum partis valorum ipſius v vinculo quadratico incluſæ. Sit $AB = z$, BDC ordinata ſecans curvam in duobus punctis, adeoque rectarum BD, BC quamlibet ambigue repreſentat ordinata v: Id eſt $BC = \frac{bz^2+cz+d}{2z+2a} + p$,

BD

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$BD = \frac{bz^2 + cz + d}{2z + 2a} - p$. Harum differentia $CD = 2p$. Biſeca *CD* in *F*, ut ſit $DF = p$, huic adde *BD*, & erit $BF = \frac{bz^2 + cz + d}{2z + 2a}$, quæ eſt ordinata Lineæ biſecantis ordinatas ad curvam terminatas: quæque in caſu præſenti eſt Hyperbola Conica. Sit ea *KF*, ejuſque Aſymptoti *GE*, *GH*; quarum hæc parallela eſt ordinatæ *DC*, propterea quod ordinata *EF* ſecat Hyperbolam in unico puncto *F*. Sume Abſciſſam novam $GE = x$, ordinatam novam *EC* vel $ED = y$; eritque $EC = CF + FE$, $ED = CF - FE$, vel more Algebraico $y = EF \pm EC$. Unde in æquatione ad curvam $2EF$ erit coefficiens ipſius *y*, ut conſtat ex natura æquationis Quadraticæ. Jam ſit *e* data quantitas, & ex naturâ Hyperbolæ erit $EF = \frac{e}{2x}$; ergo $\frac{e}{x}$ erit coefficiens ipſius *y* in æquatione curvam definiente: atque exinde æquatio neceſſario induet hanc formam $yy - \frac{ey}{x} = ax^2 + bx + c + \frac{d}{x}$ vel $xy^2 - ey = ax^3 + bx^2 + cx + d$. Q. E. D.

Caſ. 2. Quod ſi deſit terminus zv^2, æquatio erit hujuſmodi $av^2 = \overline{bz^2 + cz + d} \times v + ez^3 + fz^2 + gz + :$ & erit $BF = \frac{bz^2 + cz + d}{2a}$, quæ eſt ordinata Parabolæ

rabolæ Conicæ biſecantis ordinatas. Si quando hoc accidat, mutetur Abſciſſa z in Ordinatam v, & æquatio erit $v^3+\overline{az+b}\times v^2+\overline{cz+d}\times v = ez^2+fz+f$; tolle terminum v^3 (per *Coroll.* 4. *Prop.* 5.) & æquatio erit $\overline{az+b}\times v^2+\overline{cz+d}\times v = ez^3+fz+g$: quæ (per *Caſ.* 1.) convertitur in hanc $xy^2-ey=bx^2+cx+d$, & hæc forma continetur in priori $xy^2-ey=ax^3+bx^2+cx+d$. Q. E. D.

Caſ. 3. Si deſint termini av^2, dv, erit BF $= \frac{bz^2+cz}{cz} = \frac{bz+c}{2}$, quæ eſt ordinata rectæ biſecantis ordinatas. In illo caſu ſume abſciſſam x in rectâ illâ biſecante à debito initio computatam, & ordinatam y priori v parallelam; & æquatio induet hanc formam $xyy=ax^3+bx^2+cx+d$, quæ continetur in priori. Q. E. D.

Caſ. 4. Si deſint termini zv^2, av^2, reſtabit $\overline{bz^2+cz+d}\times v=ez^3+fz^2gz+h$, quo in caſu ordinata occurrit curvæ in unico tantum puncto. Tolle terminum ez^3 (per *Coroll.* 4. *Prop.* 5.) & æquatio erit $\overline{bz^2+cz+d}\times v=fz^2+gz+h$: hæc æquatio mutando abſciſſam z in ordinatam v, convertetur (per *Caſ.* 1.) in hanc $xy^2-ey=cx+d$, quæ continentur in formâ $xy^2-ey=ax^3+bx^2+cx+d$. Q. E. D.

Caſ. 5. Si deſint termini zv^2, av^2, bz^2v, manebit $czv+dv=ez^3+fz^2+gz+h$, ubi ſi pro v ſcribas y, & $x-\frac{d}{c}$ pro z, orietur $xy=ax^3+bx^2+cx+d$. Q. E. D.

Caſ.

Caſ. 6. Si deſint termini zv^2, bz^2v, erit $BF = \frac{cz+d}{2a}$, quæ eſt ordinata Lineæ biſecantis ordinatas CD; in illo caſu ſume Abſciſſam x in rectâ illâ biſecante, & ordinatam v priori parallelam, & prodibit $yy = ax^3 + bx^2 + cx + d$. Q. E. D.

Caſ. 7. Si deſint termini zv^2, av^2, bz^2v, czv, reſtabit $dv = ez^3 + fz^2 + gz + h$, quæ eſt hujus formæ $y = ax^3 + bx^2 + cx + d$. Q. E. D.

Reducuntur ergo omnes Lineæ tertii ordinis ad quatuor ſequentes æquationum caſus,

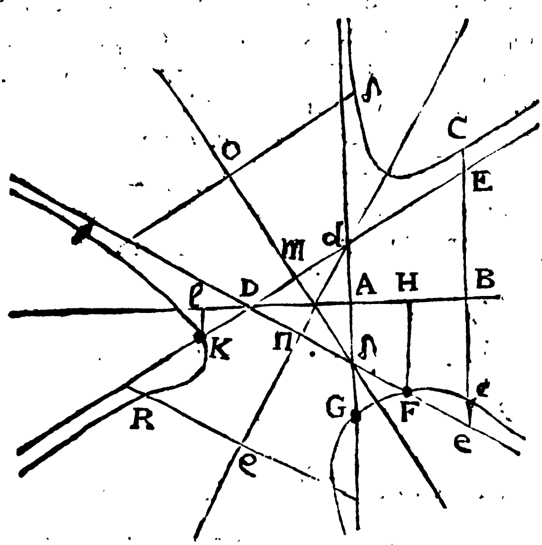
$$\left.\begin{array}{r} xy^2 - ey \\ xy \\ yy \\ y \end{array}\right\} = ax^3 + bx^2 + cx + d, \text{ ut opportebat.}$$

PROP. XVI. THEOR.

Æquatio $xy^2 - ey = ax^3 + bx^2 + cx + d$ *deſignat figuram habentem ſex crura Hyperbolica ad tres Aſymptotos jacentia.*

SIT $AB = x$, $BC = y$. Invenies $y = \frac{e}{2x} \pm \sqrt{ax^2 + bx + c + \frac{d}{x} + \frac{e^2}{4x^2}}$; reducatur (per Theor. *Neutoni*) pars irrationalis in ſeriem $\pm \frac{e}{2x} \pm \frac{d}{e}$ &c. eo citius convergentem, quo minor eſt x; at-

atque provenient ordinatæ valores $\frac{e}{x} + \frac{d}{e}$ &c. $-\frac{d}{e}$ &c. Valor hic ultimus indicat curvam ſe-



care ordinatam primam, puta in G. Valor autem ille $\frac{e}{x} + \frac{d}{e}$ &c. indicat (per *Coroll.* 2. *Prop.* 7.) ordinatam primam eſſe Aſymptoton, & habere duo crura ad diverſas ejus partes poſita, & in plagas oppoſitas tendentia. Evadat jam x utcunque magna & augebuntur ſimul ordinatæ valores ſine limite, quare curva habet alia duo crura infinita. Mutetur ſignum ipſius x, & æquatio evadet $-xy^2 - ey = -ax^3 + bx^2 - cx + d$, vel $xy^2 + ey = ax^3 - bx^2 + cx - d$, ubi patet quod

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etiamnum augeri poteſt x in infinitum & ſimul augebuntur ordinatæ valores, nunquam enim evadent impoſſibiles quum abſciſſa eſt ſatis magna? Unde Figura habet alia duo crura & omnino ſex.

Reducatur jam y in ſeriem eo citius convergentem quo major eſt x, atque invenietur $BC = x\sqrt{a} + \frac{b}{2\sqrt{a}} + \frac{4ac - bb + 4ae\sqrt{a}}{8ax\sqrt{a}} +$ &c. & $Bc = -x\sqrt{a} + \frac{b}{2\sqrt{a}} + \frac{4ac - bb - 4ae\sqrt{a}}{8ax\sqrt{a}}$ &c. Habet igitur (per *Prop.* 6.) figura duas Aſymptotos rectas ab Abſciſsâ hinc inde æqualiter jacentes: nam ſit $AD = \frac{b}{2a}$, & Ad, $A\delta$ hinc inde æquales $\frac{b}{2\sqrt{a}}$, junge Dd, $D\delta$ erunt illæ duæ Aſymptoti. Ergo Figura habet ſex crura ad tres Aſymptotos rectas jacentia. Q. E. D.

Coroll. 1. Si deſit terminus bxx, tres Aſymptoti, evaneſcente triangulo $Dd\delta$, in uno eodemque puncto conveniunt.

Coroll. 2. Si figura Ovalem conjugatam habeat, ea ſemper continetur intra triangulum $Dd\delta$; nam ſi conſiſteret extra, duci poterat recta ſecans curvam in quatuor punctis, quod fieri nequit. Idem intellige de Nodo, Cuſpide & puncto conjugato. Ut enim Punctum eſt Ovalis infinite parva, ſic Cuſpis eſt Nodus infinite parvus.

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Coroll. 3. Ergo ſi deſit terminus bx^2, hoc eſt, ſi tres Aſymptoti in uno eodemque puncto conveniunt, Figura nunquam habet Ovalem, Nodum Cuſpidem aut Punctum conjugatum: quia (per *Coroll.* 1, 2.) evaneſcit triangulum intra quod ſemper conſiſtit Ovalis, Nodus, Cuſpis vel punctum conjugatum.

Coroll. 4. Duc δMO biſecantem Dd, in M, item OP Aſymptoto Dd parallelam ſecantem curvam in P. Sitque Abſciſſa $MO=z$, Ordinata $OP=v$; & eâdem ratione, quâ exiſtente $AB=x$, $BC=y$, prodit $xy^2-ey=ax^3+bx^2+cx+d$, prodibit etiam $zv^2-Ev=Az^3+Bz^2+Cz+D$. Et ſi ducatur dNQ biſecans $D\delta$ in N, & QR Aſymptoto $D\delta$ parallela, curvam ſecans in R, & ſit $NQ=s$, $RQ=t$, orietur $st^2-\epsilon t=\alpha s^3+\beta s^2+\gamma s+\delta$. Qui de hiſce dubitat utatur calculo.

Coroll. 5. Unde ſi æquationis $xy^2-ey=ax^3+bx^2+cx+d$ deſit terminus bx^2, in æquationibus duabus reliquis deerunt termini reſpectivi Bz^2, βs^2. Et contra, ut conſtat ex *Coroll.* 1.

Coroll. 6. Si deſit terminus ey, Abſciſſa eſt Diameter biſecans Ordinatas: Et contra, ſi AB ſit Diameter, deeſt terminus ey. Simile intellige de Abſciſſis MO, NQ earumque Ordinatis.

Coroll. 7. Si deſit terminis ey, Curva non decuſſat Aſymptoton $d\delta$; nam exiſtente x infinite parva erit $y=\pm\sqrt{\frac{d}{x}}$, adeoque (per *Coroll.* 3. *Prop.* 7.) crura jacent ad eaſdem Aſymptoti $d\delta$ partes & in plagas oppoſitas feruntur: & inde (per *Coroll.* 4. *Prop.* 4.) tria interſectionis puncta

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abeunt in infinitum; adeoque nullum reſtabit punctum in quo curva decuſſare poteſt ejus Aſymptoton. Et contra, ſi curva non decuſſet Aſymptoton, deeſt terminus ey & Abſciſſa AB eſt Diameter, atque crura jacent ad eaſdem Aſymptoti $d\delta$ partes, in plagas oppoſitas lata. Simile intellige de Aſymptotis duabus reliquis cum Abſciſſis δO, dQ.

Coroll. 8. Si ſit $bb - 4ac = 4ae\sqrt{a}$, curva non decuſſat Aſymptoton Dd, adeoque (per *Coroll.* 7.) MO eſt Diameter. Nam eſt $AB = x$, $BC = y$, $BE = x\sqrt{a} + \frac{b}{2\sqrt{a}}$; quod ſi curva ſecaret Aſymptoton Dd, erit in illo caſu $BE = BC$, id eſt $y = x\sqrt{a} + \frac{b}{2\sqrt{a}}$. In æquatione pro y ſubſtitue hunc valorem, & invenies $x = \frac{4ad + 2be\sqrt{a}}{bb - 4ac - 4ae\sqrt{a}} = AL$, ubi LK eſt Ordinata tranſiens per K punctum interſectionis. Jam ſit $bb - 4ac - 4ae\sqrt{a} = 0$, vel $bb - 4ac = 4ae\sqrt{a}$, & erit AL infinite magna, adeoque punctum interſectionis nullibi erit. Similiter ſi F ſit punctum in quo curva ſecat Aſymptoton $D\delta$, & FH ordinata; erit $AH = \frac{4ad - 2bc\sqrt{a}}{bb - 4ac + 4ae\sqrt{a}}$, unde ſi ſit $bb - 4ac = -4ae\sqrt{a}$, Curva non decuſſat Aſymptoton $D\delta$, & eſt NQ Diameter.

Coroll. 9. Ergo ſi neque terminus ey deſit, neque ſit $bb - 4ac = \pm 4ae\sqrt{a}$, curva nullam habebit

bit Diametrum, ſin eorum alterutrum accidit, curva unicam habebit Diametrum & tres, ſi utrumque. Sciendum enim eſt Ordinatas biſectas eſſe alicui Aſymptoto parallelas, ut conſtat per converſum *Prop.* 5. ejuſque *Scholium*; & Diametrum ſemper biſecare Ordinatas in Aſymptotis terminatas, quia curva concidit cum Aſymptotis ad diſtantiam infinitam: adeoque Diametrum tranſire per interſectionem duarum Aſymptoton neceſſe eſt.

Coroll. 10. Igitur hæc curva vel habet nullam, unam vel tres Diametros: duas enim ſolas habere nequit.

Coroll. 11. Curva quæ nullam habet Diametrum, ſecat tres ejus Aſymptotos, ſingulam in unico puncto.

Coroll. 12. Curva quæ unicam habet Diametrum decuſſat duas Aſymptotos, per quarum interſectionem tranſit illa Diameter: at tertiam non ſecat.

Coroll. 13. Curva quæ tres habet Diametros Aſymptoton nullam omnino ſecabit.

Coroll. 14. Si $bb - 4ac - 4ae\sqrt{a}$ ſit quantitas affirmativa, Aſymptotos *DE* jacet inter curvam & Abſciſſam; ſin negativa ſit, curva jacet inter Aſymptoton & Abſciſſam. Et ſi $bb - 4ac + 4ae\sqrt{a}$ ſit affirmativa quantitas, Aſymptotos *De* jacet inter curvam & Abſciſſam, ſin negativa ſit, curva jacet inter Aſymptoton & Abſciſſam.

Coroll. 15. Si deſit terminus *ey*, id eſt, ſi *AB* ſit Diameter, & ſit $bb - 4ac$ quantitas affirmativa, curva continet Aſymptotos *Dd*, *Dδ* in ſuo ſinu,

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ſitu. At ſi quantitas illa ſit negativa, Aſymptoti jacent extra crura adjacentia.

Coroll. 16. Si figura habet tres Diametros, & ſit d quantitas affirmativa, crura jacent extra Aſymptotos; ſin minus intra. Corollaria hæc tria ultima conſtant ex *Coroll.* 2. *Prop.* 6.

Coroll. 17. Si æquationis $xy^2 - ey = ax^3 + bx^2 + cx + d$ terminus ax^3 ſit negativus, figura habebit duo tantum crura Hyperbolica ad ordinatam primam jacentia. Nam ſeries $\pm x\sqrt{a} \pm \frac{b}{2\sqrt{a}} +$ &c. ex cujus poſſibilitate pendent reliqua quatuor crura cum earum Aſymptotis erit impoſſibilis. Conſtat etiam hoc *Corollarium* ex *Exemplo tertio Scholii Prop.* 7.

Notandum eſt in hâc Propoſitione ejuſque Corollariis per Diametrum ſemper intelligi Diametrum quæ biſecat Ordinatas duarum dimenſionum.

Poſtquam compertus eſt numerus crurum alicujus curvæ, ejus ſpecies enumerantur determinando quæ crura ductu continuo conjunguntur; ut & deſcribendo Ovales, Nodos, Cuſpides & Puncta conjugata ſiquæ ſint. Hæc omnia ex Propoſitionibus præcedentibus facillime perficiuntur.

Enumeratio Specierum curvæ quam deſignat æquatio $xy^2 - ey = ax^3 + bx^2 + cx + d$.

Hyperbolæ novem ſequuntur, ſex cruribus, ad tres rectas triangulum capientes, jacentibus præditi; quæ Diametro ad Ordinatas duarum di-

dimenſionum deſtituuntur. *Vide figuras in Enumeratione* Neutonianâ *Linearum tertii ordinis.*

Si æquationis $xy^2 - ey = ax^3 + bx^2 + cx + d$, extrahatur radix y, invenietur $y = \frac{e}{2x} \pm \frac{\sqrt{ax^4 + bx^3 + cx^2 + dx + \frac{1}{4}ee}}{x}$, patet ergo ordinatam poſſibilem eſſe, quando quantitas $ax^4 + bx^3$ $cx^2 + dx + \frac{1}{4}ee$ vinculo quadratico incluſa eſt affirmativa, & impoſſibilem quando negativa: vices autem poſſibilitatis & impoſſibilitatis innoteſcunt per *Exemp.* 3. *Prop.* 8. deſcribendo Parabolam cujus abſciſſa eſt x & ordinata $ax^4 + bx^3 + cx^2 + dx + \frac{1}{4}ee$.

Species I. *Fig.* 1, 2.

Si æquationis $ax^4 + bx^3 + cx^2 + dx + \frac{1}{4}ee = 0$ radices omnes AP, $A\varpi$, $A\pi$, Ap ſint reales ejuſdem ſigni & inæquales, Figura conſtat ex tribus Hyperbolis Inſcriptâ, Circumſcriptâ & Ambigenâ. Nam (per *Exemp.* 3. *Prop.* 8.) ordinata inter puncta P, ϖ vel π, p erecta eſt imaginaria, & realis erit ordinata alio quovis abſciſſæ puncto inſiſtens. Erige ordinatas PT, $\varpi\tau$, $\pi\daleth$, pt, & hæ tangent curvam in totidem punctis T, τ, $\daleth$, t: etenim in illo caſu ordinatarum vel ſumma vel differentia evaneſcit, prout ad diverſas vel eaſdem Abſciſſæ partes extenduntur. Unde puncta T, τ, $\daleth$, t, ſunt limites poſſibilitatis & impoſſibilitatis, atque adeo intra medios limites τ, $\daleth$ continetur Ovalis. Crura vero quæ jacent ad angulum D conjunguntur: quoniam ordi-

ordinata inter *p*, *π* erecta non occurit curvæ. Sed & crura quæ jacent ad angulos *d*, *δ* etiam conjunguntur; aliter enim, ſi fieri poteſt, conjungantur, & duci poterit recta per Ovalem ſecans curvam in quatuor punctis, quod fieri nequit. Ex ſupradictis abunde patet quod crura ſemper jacent ad diverſas Aſymptoti partes; & quod Hyperbola Conica biſecat ordinatas duarum dimenſionum; ex quarum conſideratione neceſſario ſequitur Hyperbolarum unam eſſe Inſcriptam, alteram Circumſcriptam, & tertiam Ambigenam: Quæ eſt Species prima.

Species II. *Fig.* 3, 4.

Si ex radicibus duæ maximæ *Ap*, *Aπ*, (*fig.* 3.) vel duæ minimæ *AP*, *Aϖ* (*fig.* 4.) ſint æquales, Ovalis tangit Hyperbolam Circumſcriptam, & tangendo evadit Nodus, atque Hyperbola, Nodata; adeoque Figura conſtat ex tribus Hyperbolis Inſcriptâ, Nodatâ & Ambigenâ: Quæ eſt Species ſecunda.

Species III. *Fig.* 5, 6.

Si radices tres maximæ (*fig.* 5.) vel tres minimæ (*fig.* 6.) inter ſe æquentur, Nodus evadit infinite parvus, id eſt, Cuſpis, & Figura conſtat ex Hyperbolis tribus Inſcriptâ, Cuſpidatâ & Ambigenâ. Quæ eſt Species tertia.

Notandum eſt crura Hyperbolæ Nodatæ ſemper eſſe verſus ſe invicem convexa; aliter enim duci poterat recta ſecans curvam in quatuor punctis. Idem intellige de Cuſpidatâ, ſiquidem

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quidem Cuſpis nihil aliud eſt quam Nodus infinite parvus.

Species IV. *Fig.* 7.

Si è radicibus duæ mediæ æquentur (*fig.* 7.) Ovalis, quæ in ſpecie primâ obtinebat, evadit infinite parva, id eſt, Punctum. Et Figura conſtat ex Hyperbolis tribus Inſcriptâ, Circumſcriptâ & Ambigenâ cum Puncto conjugato: Quæ eſt Species quarta.

Species V, VI. *Fig.* 7, 8, 9, 10.

Si è radicibus duæ mediæ ſint impoſſibiles, & reliquæ duæ inæquales & ejuſdem ſigni (nam contraria habere nequeunt) impoſſibile erit itidem curvam habere Ovalem, Nodum, Cuſpidem aut Punctum conjugatum; adeoque Figura erit pura conſtans ex Hyperbolis tribus Inſcriptâ, Circumſcriptâ & Ambigenâ. Si hæ Hyperbolæ jaceant ad angulos trianguli *Ddd*, (*fig.* 7, 8.) Species eſt quinta; ſin jaceant ad latera ejuſdem (*fig.* 9, 10.) Species eſt ſexta.

Species VII, VIII. *Fig.* 11, 12, 13, 14.

Si è radicibus duæ ſint æquales, & alteræ duæ vel impoſſibiles (*fig.* 11, 13.) vel poſſibiles (*fig.* 12, 14.) cum ſignis quæ à ſignis æqualium radicum diverſa ſunt, quatuor crura in uno puncto conveniunt; ſcilicet Hyperbolæ quæ in ſpeciebus quintâ & ſextâ erant circumſcriptæ & ambigenæ nunc conſtituunt Cruciformem. Adeoque Figura conſtat ex Inſcripta & Cruciformi.

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formi. Quod ſi jaceat Inſcripta ad angulum trianguli *Ddδ* (*fig.* 11, 12.) Species eſt ſeptima. At ſi jaceat ad latus (*fig.* 13, 14.) Species eſt octava.

Species IX. *Fig.* 15, 16.

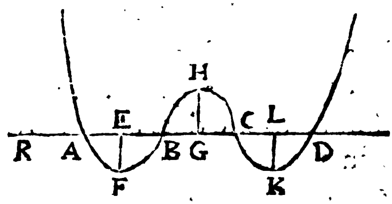
Si radices omnes ſint impoſſibiles (*fig.* 15.) vel ſi omnes ſint reales (*fig.* 16.) & earum duæ negativæ ſint & alteræ duæ affirmativæ, Hyperbolæ quæ in ſpeciebus ſeptimâ & octavâ conjungebantur & conſtituebant Cruciformen ab invicem iterum ſeperantur; & Figura conſtat ex Hyperbolis duabus Inſcriptis in angulis Aſymptoton oppoſitis jacentibus, cum Anguinea circa tertiam Aſymptoton flexâ: Quæ eſt Species nona.

Si radices duæ æquentur & duæ reliquæ etiam æquentur, figura migrat in ſectionem Conicam cum lineâ rectâ. Nam $\sqrt{ax^4+bx^3+cx^2+dx+\frac{1}{4}ee}$ erit quantitas rationalis, & inde æquatio $xy^2-ey=ax^3+bx^2+cx+d$ bipartitur in duas æquationes, quarum una deſignat Hyperbolam Conicam, altera rectam.

Hi ſunt caſus omnes poſſibiles radicum æquationis $ax^4+bx^3+cx^2+dx+\frac{1}{4}ee$ propterea quod $\frac{1}{4}ee$ terminus ultimus eſt affirmativus, quippe quadratum realis quantitatis. Ut vero hoc melius intelligatur, caſus unius impoſſibilitatem oſtendam, cujus ad Exemplum reliquorum impoſſibilitas facillime evincitur.

Si æquationis $ax^4 + bx^3 + cx^2 + dx + \frac{1}{4}ee = 0$, radices tres idem habeant ſignum, dico quartum diverſum habere non poſſe.

Deſcribatur Parabola æquatione $z = ax^4 + bx^3$ $cx^2 + dx + \frac{1}{4}e^2$ deſignata; quoniam æquationis



$ax^4 + bx^3 + cx^2 + dx + \frac{1}{4}ee = 0$ quatuor radices ponuntur reales, Abſciſſa ſecat curvam in quatuor punctis: ea ſunto *A*, *B*, *C*, *D*. Jam ſi fieri poſſit, ſit principium Abſciſſæ inter puncta *A*, *B* adeo ut radix *EA* ſit negativa & reliquæ tres affirmativæ: ſit *EF* Ordinata prima; & exiſtente $x = 0$, erit $y = \frac{1}{4}ee = EF$: Sed *EF* eſt quantitas negativa, ergo etiam $\frac{1}{4}ee$ quantitas negativa, quod eſt abſurdum. Similiter oſtendetur quod Principium Abſciſſæ jacere nequit inter puncta *C*, *D*, quoniam ordinatæ ad Abſciſſæ partem *CD* erectæ ſunt omnes negativæ. Igitur conſtat propoſitum; nam ſi tres radices eadem habent ſigna, & quarta diverſum: principium Abſciſſæ vel erit inter puncta *C*, *D* vel *A*, *B*. Sed cum hoc fieri nequit, neque illud fieri poteſt.

Eodemque modo oſtenditur, quod ſi æquationis $ax^4 + bx^3 + cx^2 + dx + \frac{1}{4}ee = 0$ radices duæ ſint impoſſibiles, reliquæ duæ idem ſignum habebunt.

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Hyperbolæ quatuordecem cum ſex cruribus ad tres Aſymptotos triangulum capientes jacentibus, unicam habentes Diametrum ad Ordinatas duarum dimenſionum.

Si deſit terminus ey Figura habet Diametrum ad ordinatas duarum dimenſionum, ſcilicet Abſciſſam; & æquatio erit $xy^2 = ax^3 + bx^2 + cx + d$.

Species X, XI. *Fig.* 17.

Si æquationis $ax^3 + bx^2 + cx + d = 0$ radices omnes ſint reales ejuſdem ſigni & inæquales, Figura conſtat ex tribus Hyperbolis ad tres angulos trianguli $Dd\delta$ jacentibus cum Ovali intra triangulum conſiſtenti. Nam ſint tres radices $A\tau$, $A\daleth$, At, & erunt T, $\daleth$, t limites poſſibilitatis & impoſſibilitatis (*per Exemp.* 2. *Prop.* 8.) atque intra τ, t continetur Ovalis. Crura vero, quæ ad eoſdem angulos trianguli $Dd\delta$ jacent conjungi, eodem modo oſtenditur, quo in ſpecie primâ. Si Aſymptoti Dd, $D\delta$ jaceant intra crura (*fig.* 17.) curva conſtat ex tribus Hyperbolis, duabus Inſcriptis ad d, δ & circumſcriptâ ad D: Quæ eſt Species decima.

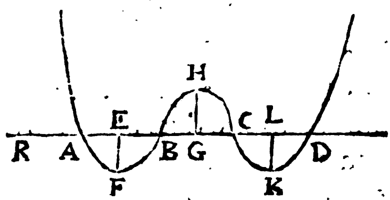
Sin Aſymptotos jaceant extra crura, Figura conſtat ex tribus Hyperbolis, Inſcriptâ ad D, & Ambigenis duabus ad d, δ: Quæ eſt Species undecima.

Species XII. *Fig.* 18.

Si radices duæ maximæ At, $A\daleth$ æquentur, Ovalis tangit Hyperbolam Circumſcriptam, & tangendo evadit Nodus, atque Hyperbola Nodata;

Si æquationis $ax^4+bx^3+cx^2+dx+\frac{1}{4}ee=0$, radices tres idem habeant ſignum, dico quartum diverſum habere non poſſe.

Deſcribatur Parabola æquatione $z=ax^4+bx^3$ $cx^2+dx+\frac{1}{4}e^2$ deſignata; quoniam æquationis



$ax^4+bx^3+cx^2+dx+\frac{1}{4}ee=0$ quatuor radices ponuntur reales, Abſciſſa ſecat curvam in quatuor punctis: ea ſunto *A*, *B*, *C*, *D*. Jam ſi fieri poſſit, ſit principium Abſciſſæ inter puncta *A*, *B* adeo ut radix *EA* ſit negativa & reliquæ tres affirmativæ: ſit *EF* Ordinata prima; & exiſtente $x=0$, erit $y=\frac{1}{4}ee=EF$: Sed *EF* eſt quantitas negativa, ergo etiam $\frac{1}{4}ee$ quantitas negativa, quod eſt abſurdum. Similiter oſtendetur quod Principium Abſciſſæ jacere nequit inter puncta *C*, *D*, quoniam ordinatæ ad Abſciſſæ partem *CD* erectæ ſunt omnes negativæ. Igitur conſtat propoſitum; nam ſi tres radices eadem habent ſigna, & quarta diverſum: principium Abſciſſæ vel erit inter puncta *C*, *D* vel *A*, *B*. Sed cum hoc fieri nequit, neque illud fieri poteſt.

Eodemque modo oſtenditur, quod ſi æquationis $ax^4+bx^3+cx^2+dx+\frac{1}{4}ee=0$ radices duæ ſint impoſſibiles, reliquæ duæ idem ſignum habebunt.

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Hyperbolæ quatuordecem cum ſex cruribus ad tres Aſymptotos triangulum capientes jacentibus, unicam habentes Diametrum ad Ordinatas duarum dimenſionum.

Si deſit terminus ey Figura habet Diametrum ad ordinatas duarum dimenſionum, ſcilicet Abſciſſam: & æquatio erit $xy^2 = ax^3 + bx^2 + cx + d$.

Species X, XI. *Fig.* 17.

Si æquationis $ax^3 + bx^2 + cx + d = 0$ radices omnes ſint reales ejuſdem ſigni & inæquales, Figura conſtat ex tribus Hyperbolis ad tres angulos trianguli $Dd\delta$ jacentibus cum Ovali intra triangulum conſiſtenti. Nam ſint tres radices $A\tau$, $A\daleth$, At, & erunt T, $\daleth$, t limites poſſibilitatis & impoſſibilitatis (*per Exemp.* 2. *Prop.* 8.) atque intra τ, t continetur Ovalis. Crura vero, quæ ad eoſdem angulos trianguli $Dd\delta$ jacent conjungi, eodem modo oſtenditur, quo in ſpecie primâ. Si Aſymptoti Dd, $D\delta$ jaceant intra crura (*fig.* 17.) curva conſtat ex tribus Hyperbolis, duabus Inſcriptis ad d, δ & circumſcriptâ ad D: Quæ eſt Species decima.

Sin Aſymptotos jaceant extra crura, Figura conſtat ex tribus Hyperbolis, Inſcriptâ ad D, & Ambigenis duabus ad d, δ: Quæ eſt Species undecima.

Species XII. *Fig.* 18.

Si radices duæ maximæ At, $A\daleth$ æquentur, Ovalis tangit Hyperbolam Circumſcriptam, & tangendo evadit Nodus, atque Hyperbola Nodata;

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datâ: Et Figura conſtat ex Hyperbolis duabus Inſcriptis ad *d*, *δ* cum Nodatâ ad *D*: Quæ eſt Species decima ſecunda.

Species XIII. *Fig.* 19.

Si radices tres æquentur, Nodus evadit Cuſpis, & curva conſtat ex Hyperbolis duabus Inſcriptis ad *d*, *δ*, & Cuſpidata ad *D*: Quæ eſt Species decima tertia.

Species XIV, XV. *Fig.* 20, 21.

Si radices duæ minimæ æquentur; Ovalis, quæ in ſpeciebus 10ma, & 11ma, obtinebat evadit Punctum. Et Figura vel conſtat ex Hyperbolis duabus Inſcriptis ad *d*, *δ* (*fig.* 20.) cum Circumſcriptâ ad *D*: Quæ eſt decima quarta; vel conſtat ex Hyperbolis duabus Ambigenis ad *d*, *δ* (*fig.* 21.) cum Inſcriptâ ad *D*: Quæ eſt Species decima quinta.

Species XVI, XVII, XVIII, XIX.
Fig. 20, 21, 22, 23.

Si ex radicibus duæ ſint impoſſibiles, puræ habebuntur tres Hyperbolæ ſine Ovali, Decuſſatione, Cuſpide vel Puncto conjugato. Et hujus ſpecies ſunt quatuor: nempe decima ſexta (*fig.* 20.) ſi Circumſcripta jaceant ad *D*, decima ſeptima (*fig.* 21.) ſi Inſcripta jaceat ad *D*; decima octava (*fig.* 22.) ſi circumſcripta jaceat ad latus trianguli, & decima nona (*fig.* 23.) ſi Inſcripta jaceat ad latus trianguli.

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Species XX, XXI. *Fig.* 25, 26.

Si radices ſint æquales, & tertia eſt ſigni diverſi, quatuor crura in Abſciſsâ concurrunt. Et Figura conſtabit ex Inſcriptâ & Cruciformi. Quod ſi Inſcripta (*fig.* 25.) jaceat ad angulum trianguli, Species eſt vigeſima; ſin jaceat Inſcripta (*fig.* 24.) ad latus trianguli, Species eſt vigeſima prima.

Species XXII, XXIII. *Fig*, 26, 27.

Si radices duæ ſint inæquales & tertia eſt ſigni diverſi, Crura quatuor quæ in ſpeciebus duabus noviſſimis conjungebantur ab invicem ſegregantur. Et Figura conſtabit ex Hyperbolis duabus Inſcriptis, in angulis Aſymptoton oppoſitis jacentibus cum Conchoide intermediâ. Si Conchois (*fig.* 24.) jaceat ad eaſdem Aſymptoti *dd* partes cum triangulo *Ddδ*, Species eſt vigeſima ſecunda; ſin jaceat ad diverſas (*fig.* 25.) Species eſt vigeſima tertia. Hi ſunt caſus omnes radicum æquationis $xy^2 = ax^3 + bx^2 + cx + d$.

Hyperbolæ quatuor tum ſex cruribus ad tres Aſymptotos triangulum facientes jacentibus, quæ tres habent Diametros ad ordinatas duarum dimenſionum.

Si in æquatione $xy^2 = ax^3 + bx^2 + cx + d$, ſit $bb = 4ac$, Figura habet tres Diametros ad ordinatas duarum dimenſionum: & crura eandem Aſymptoton adjacentia ad eaſdem ejus partes ſemper in plagas oppoſitas protenſa jacent. Nunquam

quam hæc Figura decuſſat Aſymptotos, adeoque ſemper conſtat ex tribus Inſcriptis Hyperbolis. Sequitur (per *Coroll.* 9. *Prop.* 16.)

Species XXIV. *Fig.* 28.

Si æquationis $ax^3+bx^2+\frac{bb}{4a}x+d=0$ (poſito pro c ejus valore $\frac{bb}{4a}$) radices tres ſint reales, ejuſdem ſigni & inæquales, Figura conſtat ex Hyperbolis tribus Inſcriptis cum Ovali intra triangulum $Dd\delta$: Quæ eſt Species vigeſima quarta.

Species XXV. *Fig.* 28.

Si radices duæ minimæ æquentur, Ovalis in punctum evaneſcet, & Figura conſtat ex tribus Hyperbolis Inſcriptis cum Puncto conjugato: Quæ eſt Species vigeſima quinta.

Species XXVI, XXVII. *Fig.* 28, 29.

Si radices duæ imaginariæ ſint, Figura conſtat ex tribus Hyperbolis Inſcriptis ſine Ovali vel puncto conjugato. Quod ſi hæ Hyperbolæ jaceant ad angulos trianguli $Dd\delta$ (*fig.* 28.) Species eſt vigeſima ſexta, ſin jaceant (*fig.* 29.) ad latera ejuſdem, Species eſt vigeſima ſeptima.

Hi ſunt omnes caſus radicum æquationis $ax^3+bx^2+\frac{bb}{4a}x+d=0$; nam impoſſibile eſt duas

ejus

ejus radices maximas, vel omnes inter ſe æquari. Impoſſibile itidem eſt duas radices eſſe ejuſdem ſigni, & tertiam ſigni ab iis diverſi.

Hyperbolæ novem cum ſex cruribus jacentibus ad tres Aſymptotos ad unum punctum convergentes.

Si æquationis $xy^2 - ey = ax^3 + bx^2 + cx + d$ deſit terminus bx^2, tres Aſymptoti (per *Coroll.* 1. *Prop.* 16.) in uno eodemque puncto conveniunt. Et in illo caſu (per *Coroll.* 3. *Prop.* 16.) Figura nunquam habet Ovalem, Nodum, Cuſpidem vel Punctum conjugatum: Species ejus ergo omnes ſunt puræ, & ex puris hactenus enumeratis facillime enumerantur: ut ſequitur.

Vertuntur Species quinta & ſexta in vigeſimam octavam, (*fig.* 30.)

Septima & octava in vigeſimam nonam (*fig.* 31.)

Nona in triceſimam (*fig.* 33.) quando Anguinea tranſit per centrum, & in triceſimam primam (*fig.* 32.) ubi non tranſit per centrum.

Hæ quatuor Species Diametrum non habent.

Vertuntur ſpecies decima ſexta & decima octava (*fig.* 34.) in triceſimam ſecundam.

Species decima ſeptima & decima nona (*fig.* 35.) in triceſimam tertiam.

Vigeſima & vigeſima prima (*fig.* 36.) in triceſimam quartam.

Vigeſima ſecunda & vigeſima tertia (*fig.* 37.) in triceſimam quintam.

Et

Et hæc quatuor Species unicam habent Diametrum.

Ac denique vertitur Species vigeſima ſexta & vigeſima ſeptima (*fig.* 38.) in triceſimam ſextam. Hæc Figura habet tres Diametros.

Nulla hic oriri poteſt difficultas, modo conſideremus ſpecierum ſingularum duarum, in unam hic coaleſcentium, diverſitatem antea debitam fuiſſe triangulo ab Aſymptotis comprehenſo, quod nunc in nihil evanuit.

Hyperbolæ ſex cum duobus cruribus ad eandem Aſymptoton jacentibus.

Si æquationis $xy^2 - ey = ax^3 + bx^2 + cx + d$ terminus ax^3 ſit negativus, Figura (per *Coroll.* 17. *Prop.* 16.) habebit duo tantum crura ad unicam Aſymptoton jacentia: & hæc crura (per *Coroll.* 11. *Prop.* 16.) ad diverſas Aſymptoti partes in plagas oppoſitas extenſa jacebunt. In iſto caſu erit $y = \frac{\frac{1}{2}e \pm \sqrt{-ax^4 + bx^3 + cx^2 + dx + \frac{1}{4}e^2}}{x}$ ubi conſtat ordinatam y fore poſſibilem quando quantitas $-ax^4 + bx^3 + cx^2 + dx + \frac{1}{4}e^2$ eſt affirmativa, & impoſſibilem quando negativa.

Species XXXVII. *Fig.* 39.

Si æquationis $ax^4 = bx^3 + cx^2 + dx + \frac{1}{4}ee$ radices omnes ſint reales & inæquales, eæ ſunto AP, $A\varpi$, $A\pi$, Ap; erige ordinatas PT, $\varpi\tau$, $\pi\daleth$, pt, & erunt T, τ, $\daleth$, t limites; & intra τ, t continetur Ovalis: adeoque Figura conſtat ex Ovali cum An-

Anguinea circa Aſymptoton flexâ: Quæ eſt Species triceſima ſeptima.

Species XXVIII. *Fig.* 40.

Si radices duæ mediæ *AP*, *Ap* æquentur Ovalis tangit Anguineam, & inde Figura conſtat ex Hyperbolâ unica Nodata: Quæ eſt Species triceſima octava.

Species XXXIX. *Fig.* 41.

Si radices tres ejuſdem ſigni ſint æquales, Nodus migrat in Cuſpidem, & Figura conſtat ex unicâ Cuſpidatâ: Quæ eſt Species triceſima nona.

Species XL. *Fig.* 43.

Si è radicibus ejuſdem ſigni duæ maximæ æquentur; Ovalis quæ in ſpecie 37ma. obtinebat, in punctum minuetur: & Figura conſtat ex Puncto cum Anguineâ circa ordinatam primam flexâ: Quæ eſt Species quadrageſima.

Species XLI, XLII. *Fig.* 42, 43.

Si è radicibus duæ ſint impoſſibiles, manebit Anguinea pura: Quod ſi illa tranſit per centrum (*fig.* 43.) Species eſt quadrageſima prima.

At ſi non tranſit per centrum (*fig.* 42.) Species eſt quadrageſima ſecunda.

Hyperbolæ ſeptem cum duobus cruribus ad eandem Aſymptoton extenſis cum Diametro ad Ordinatas duarum dimenſionum.

Si deſit terminus *ey*, Abſciſſa eſt Diameter, & (per *Coroll.* 7. *Prop.* 16.) crura ad eaſdem

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ordinatæ primæ partes jacentia in plagas contrarias in infinitum ſerpunt. Invenietur $y = \pm \frac{\sqrt{-ax^4+bx^3+cx^2+dx}}{x}$: adeoque omnes caſus æquationis pateſcunt deſcribendo Parabolam, æquatione $z = -ax^4+bx^3+cx^2+dx$ definitam.

Species XLIII. *Fig.* 44.

Si æqations $ax^3 = bx^2+cx+d$ radices ſint omnes reales ejuſdem ſigni & inæquales, Figura conſtat ex Conchoide cum Ovali ad convexitatem ejus: Quæ eſt Species quadrageſima tertia.

Species XLIV. *Fig.* 45.

Si radices duæ ſint inæquales, & ejuſdem ſigni, & tertia contrarii, Figura conſtat ex Conchoide cum Ovali ad concavitatem: Quæ eſt Species quadrageſima quarta.

Species XLV. *Fig.* 46.

Si radices omnes ſint ejuſdem ſigni, & duæ minimæ æquentur, Figura conſtat ex Hyperbola Nodatâ: Quæ eſt Species quadrageſima quinta.

Species XLVI. *Fig.* 47.

Si tres radices æquentur, Nodus evadit Cuſpis, & Figura evadit Ciſſois Veterum: Quæ eſt Species quadrageſima ſexta.

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Species LXVII. *Fig.* 49.

Si radices duæ maximæ æquentur & tertia ſit ejuſdem ſigni, Figura conſtat ex Conchoide cum Puncto ad convexitatem: Quæ eſt Species quadrageſima ſeptima.

Si radices duæ æquentur & tertia ſit ſigni contrarii, Figura conſtat ex Conchoide cum Puncto ad concavitatem (*fig.* 49.) Quæ eſt Species quadrageſima octava.

Species XLIX. *Fig.* 48, 49.

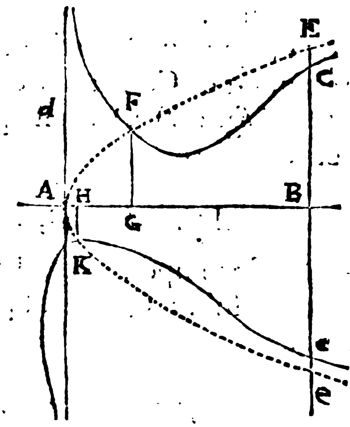
Si radices duæ ſint impoſſibiles, Figura conſtat ex Conchoide ſolâ: Quæ eſt Species quadrageſima nona.

PROP. XVII. THEOR.

ÆQuatio $xy^2 - ey = * bx^2 + cx + d$, deſignat Figuram habentem quatuor crura quorum duo ſunt Hyperbolica ad Ordinatam primam jacentia; & duo ſunt Parabolica in eandem plagam extenſa, quæ pro Aſymptoto ſortiuntur Parabolam *Conicam*.

Sit $AB = x$, $BC = y$. Invenies $y = \frac{\frac{1}{2}e}{x} \pm \frac{\sqrt{bx^3 + cx^2 + dx + \frac{1}{4}ee}}{x}$: reducatur pars irrationalis in ſeriem $\frac{e}{2x} + \frac{d}{e}$ &c. eo citius convergentem quo minor eſt x; atque unus ordinatæ va-

lor erit $-\frac{d}{e}$ &c. alter $\frac{e}{x}+\frac{d}{e}$ &c. valor ille primus indicat ordinatam primam ſecare curvam: & valor hic ultimus $\frac{e}{x}+\frac{d}{e}$ &c. (per



Coroll. 2. *Prop.* 7.) indicat ordinatam primam *Ad* eſſe Aſymptoton, habentem duo crura ad diverſas ejus partes jacentia, in plagas oppoſitas progredientia. Evadat jam x utcunque magna, & etiam ſine limite augebuntur Ordinatæ y duo valores: quæ curva habet alia duo crura infinita: Quod ſi mutetur ſignum ipſius x, hoc eſt, ſi ſumatur Abſciſſa ad alteras partes, terminus bx^2 evadet negativus; & inde augendo x ad illas partes Ordinata tandem evadet imaginaria. Igitur curva habet quatuor tantum crura infinita. Reducatur y in ſeriem $\pm\sqrt{bx}\pm\frac{c}{2\sqrt{bx}}$ &c. Unde (per *Prop.* 6.) $\pm\sqrt{bx}$ eſt Ordinata Aſym-

Aſymptoti quæ quidem eſt Parabola Conica, axem habens AB, Latus rectum b, & verticem A. Q. E. D.

Coroll. 1. Si terminus ex ſit affirmativus, Parabola Conica jacet intra crura hujus Figuræ; ſin negativus Parabola Conica Figuræ crura in ſinu ſuo complectitur: ut conſtat ex *Coroll.* 2. *Prop.* 6.

Coroll. 2. Parabola hæc Conica nunquam ſecat curvam in pluribus punctis quam in duobus. Occurrat ordinata BC parabolæ E, & erit $BE = \pm\sqrt{bx}$, ideoque ubi Parabola ſecat curvam evadit $y = \pm\sqrt{bx}$, coeuntibus punctis C, E: & $yy = bx$, unde $x = \frac{yy}{b}$ quem valorem pro x in æquatione ſubſtitue, & invenietur $y = \frac{be \pm \sqrt{b^2e^2 - 4bcd}}{2c}$: puncta igitur interſectionis ad ſummum duo tantum ſunt, quum æquatio eadem determinans eſt duarum dimenſionum.

Coroll. 3. Igitur Parabola vel ſecat Figuram in duabus punctis vel in nullis; propterea quod æquatio Quadratica vel habet duas radices poſſibiles vel nullas.

Coroll. 4. Si $4cd$ majus ſit be^2, Parabola non ſecat curvam, quoniam æquatio $y = \frac{be \pm \sqrt{b^2e^2 - 4bcd}}{2c}$ puncta interſectionis determinans erit impoſſibilis. Nam quantitas vinculo Quadratico inclusa

cluſa eſt negativa. Et in illo caſu crura Parabolæ jacent intra crura Figuræ.

Coroll. 5. Si $4cd$ æqalis ſit be^2, Parabola tangit curvam. Nam ſint *F*, *K* puncta interſectionis *FG*, *KH* ordinatæ ad Abſciſſam ; poſſunt hæ ordinatæ ad diverſas vel eaſdem Abſciſſæ partes jacere : & ſi ſit $4cd = be^2$ iſtarum rectarum evaneſcit differentia ; adeoque coeunt interſectionis puncta, & ex iis coeuntibus conflatur punctum contactûs.

Coroll. 6. Si Parabola ſecet curvam in duobus punctis, jacent hæc puncta ad eaſdem vel diverſas Abſciſſe partes, prout terminus ex eſt affimativus aut negativus : ut ex *Corollario* primo facile colligitur.

Coroll. 7. Si deſit terminus ey, Abſciſſa eſt Diameter, & curva non decuſſat Aſymptoton *Ad*, ſed crura ad eaſdem ejus partes in plagas oppoſitas extenſa jacebunt.

Coroll. 8. Et quando deeſt terminus ille ey, puncta interſectionis *FK* in eâdem ordinata jacebunt, coeuntibus punctis *G*, *H*.

Coroll. 9. Et in illo caſu Parabola ſecat curvam, vel non ſecat, prout terminus cx eſt negativus aut affirmativus : ut conſtat ex *Corollario* ſecundo.

Enumerato Specierum curvæ quam deſignat æquatio $xy^2 - ey = * \, bx^2 + cx + d.$

Figuræ ſeptem partim Hyperbolicæ, partim Parabolicæ, ſcilicet quæ habent duo crura Hyperbolica, & duo Parabolica.

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Species L. *Fig.* 50.

Si æquationis $bx^3+cx^2+dx+\frac{1}{4}ee=0$ extrahantur radices tres AP, $A\varpi$, $A\pi$, & ab earum extremis erigantur Ordinatæ totidem PT, $\varpi\tau$, $\pi\daleth$, hæ tangent curvam in punctis T, τ, $\daleth$, qui ſunt limites. Nam ſi tres radices ſint omnes reales ejuſdem ſigni & inæquales, crura Hyperbolica & Parabolica ad eaſdem Abſciſſæ partes poſitæ conjunguntur, & intra limites τ, $\daleth$, continetur Ovalis: Quæ eſt Species quinquageſima.

Species LI. *Fig.* 51.

Si è tribus radicibus ejuſdem ſigni duæ minores inter ſe æquentur, Ovalis accedet ad unam figurarum Hyperbolico-Parabolicarum Nodum efficiens: Quæ eſt Species quinquageſima prima.

Species LII. *Fig.* 52.

Si tres radices æquentur, Nodus migrabit in Cuſpidem; quæ eſt Species quinguageſima ſecunda.

Species LIII. *Fig.* 53.

Si è tribus radicibus ejuſdem ſigni, duæ majores æquentur, Ovalis evaneſcet in Punctum conjugatum: Quæ eſt Species quinquageſima tertia.

Species LIV. *Fig.* 54, 53.

Si duæ radices ſint imaginariæ: manebunt duæ puræ figuræ: Quæ eſt Species quinquageſima quarta.

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Species LV. *Fig.* 55.

Si radices duæ æquentur, & tertia eſt ſigni contrarii; Figura evadit Cruciformis quæ eſt Species quinquageſima quinta.

Species LVI. *Fig.* 56.

Si radices duæ ſint inæquales, & tertia eſt ſigni contrarii; Figurâ conſtat ex Anguinea circa Ordinatam primam flexâ, cum Parabolâ: Quæ eſt Species quinquageſima ſexta.

Figuræ quatuor Hyperbolo-Parabolicæ cum Dia-Diametro Abſciſsâ.

Species LVII. *Fig.* 57.

Si æquationis $bx^2+cx+d=0$ radices ſint impoſſibiles, crura Parabolica junguntur cum Hyperbolicis ad eaſdem Abſciſſæ partes: Quæ eſt Species quinquageſima ſeptima.

Species LVIII. *Fig.* 58.

Si radices duæ ſint æquales & ejuſdem ſigni, Figura evadit Cruciformis: Quæ eſt Species quinquageſima octava.

Species LIX. *Fig.* 59.

Si radices ſint ejuſdem ſigni & inæquales, Figura conſtat ex Conchoide cum Parabola ad eaſdem partes Ordinatæ primæ: Quæ eſt Species quinquageſima nona.

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Species LX. *Fig.* 60.

Si radices ſint ſigni diverſi, Conchois & Parabola ad diverſas ordinatæ primæ partes jacebunt: Quæ eſt Species ſexageſima.

PROP. XVIII. THEOR.

Vid. Fig. 61, 62, 63, 64.

Æquatio $xy^2 - ey = * * cx + d$ deſignat Figuram habentem ſex crura Hyperbolica ad tres Aſymptotos, quarum duæ ſunt Abſciſſæ parallelæ, jacentia.

Reducatur y in ſeriem $\frac{e}{x} + \frac{d}{e}$ &c. eo citius convergentem, quo minor eſt x: Unde (per *Cor.* 2. *Prop.* 7.) ordinata prima eſt Aſymptotos habens duo crura ad diverſas ejus partes jacentia, & in plagas oppoſitas progredientia. Abſciſſa x ad utraſque partes in infinitum augeri poteſt, & ordinata nunquam evadet impoſſibilis. Reducatur y in ſeriem eo citius convergentem quo major eſt x, $\pm\sqrt{c} + \frac{\pm d + e\sqrt{c}}{2x\sqrt{c}}$ &c. Unde (per *Prop.* 6.) $\pm\sqrt{c}$ eſt ordinata Aſymptoti ad Abſciſſam x pertinens. Sume igitur in ordinatâ primâ duas rectas hinc inde æquales $\sqrt{c}$, & rectæ ductæ per earum extremitates Abſciſſæ parallelæ, erunt duæ Aſymptoti.

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Coroll. 1. Si $d+e\sqrt{c}$ ſit quantitas affirmativa, Aſymptotos *dg* jacet inter ejus crus & Abſciſſam; ſin negativa contrarium accidit.

Coroll. 2. Crura adjacentia Aſymptotos *dg*, *δγ* ſemper jacent ad diverſas earum partes: Nam ſi ſignum Abſciſſæ mutetur, quantitatis $dt+e\sqrt{c}$ ſignum mutabitur.

Coroll. 3. Curva non decuſſat Aſymptotos *dg*, *δγ* (per *Coroll.* 6. *Prop.* 4.)

Coroll. 4. Eodem modo oſtenditur, quo in *Coroll.* 1. quod ſi ſit $d-e\sqrt{c}$ quantitas affirmativa, Aſymptotos *δγ* jacet inter crus & Abſciſſam.

Coroll. 5. Ergo ſi $d+e\sqrt{c}$, $d-e\sqrt{c}$ ſint quantitates ejuſdem ſigni, Aſymptoton *dg*, *δγ* extremitates unæ ad eandem plagam tendentes jacent extra crura, & reliquæ intra.

Coroll. 6. Si $d+e\sqrt{c}$, $d-e\sqrt{c}$ ſint ſigni contrarii, Aſymptoton *dg*, *δγ* extremitates ad plagas oppoſitas ductæ jacebunt intra crura, reliquæ extra.

Coroll. 7. Crura adjacentia Aſymptoton *dδ*, jacent ad eaſdem vel diverſas ejus partes, prout abeſt vel adeſt terminus *ey*.

Coroll. 8. Si deſit terminus *ey*, extremitates unæ Aſymptoton in eandem plagam extenſæ jacent intra crura, reliquæ extra.

Coroll. 9. Si terminus *cx* ſit negativus; Figura habet duo tantum crura ad ordinatæ primæ eaſdem vel contrarias partes jacentia, prout abeſt vel adeſt terminus *ey*.

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Enumeratio Specierum curvæ quam deſignat æquatio $xy^2 - ey = cx + d$.

Species LXI. *Fig.* 61.

Si æquationis $cx^2 + dx + \frac{1}{4}ee = 0$, radices ſint reales, eæ neceſſario habebent eadem ſigna, & erunt inæquales: atque Figura conſtabit ex tribus Hyperbolis, Inſcriptâ ad *d*, Ambigenâ ad *δ*, cum tertiâ intra Aſymptotos parallelas: Quæ eſt Species ſexageſima prima.

Species LXII, LXIII. *Fig.* 62, 63.

Si æquationis illius radices ſint imaginariæ, habebitur Anguinea intra Aſymptotos parallelas, cum duabus Inſcriptis: ſi Anguinea tranſit per centrum (*fig.* 63.) Species eſt ſexageſima ſecunda; ſi non tranſeat per centrum (*fig.* 62.) Species eſt ſexageſima tertia.

Species LXIV. *Fig.* 64

Si deſit terminus *ey*, Figura conſtat ex Hyperbolâ intra Aſymptotos parallelas, cum duabus Inſcriptis: Quæ eſt Species ſexageſima quarta.

Species LXV, LXVI. *Fig.* 65, 66.

Si æquationis $xy^2 + ey = cx + d$, terminus cx ſit negativus; Figura conſtat ex Anguineâ pura: ſi Anguinea illa tranſit per centrum, Species eſt ſexageſima quinta; at ſi non tranſit per centrum, Species eſt ſexageſima ſexta.

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Species LXVII. *Fig.* 67.

Si deſit terminus *ey*, æquatio $xy^2 = -cx + d$, deſignat Conchoidem puram: Quæ eſt Species ſexageſima ſeptima.

PROP. XIX. THEOR.

Æquatio $xy^2 - ey = * * * + d$ *deſignat Figuram habentem quatuor crura.*

VIde *Fig.* 68. ubi $AB = x$, $BC = y$, invenietur $y = \frac{\frac{1}{2}e \pm \sqrt{dx + \frac{1}{4}ee}}{x}$, id eſt per methodum ſerierum, $y = \frac{e}{x} + \frac{d}{e} +$ &c. Unde Ordinata prima *AG* eſt Aſymptotos habens duo crura adjacentia. Augeatur jam *x* perpetuo, & *y* non evadet impoſſibilis; quare curva habet alia duo crura: quod ſi mutetur ſignum ipſius *x*, evadet *y* tandem imaginaria; adeoque curva ad illas plagas in infinitum non pergit. Quoniam vero augendo *x*, *y* perpetuo diminuitur, patet Abſciſſam eſſe alteram Aſymptoton habentem duo crura ad diverſas ejus partes jacentia, in plagam eandem protenſa. Q. E. D.

Coroll. Abſciſſa non ſecat curvam; nam (per *Coroll.* 4. *Prop.* 4.) tria interſectionis puncta abeunt in infinitum.

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Species LXVIII. *Fig.* 68.

Si adſit terminus *ey*, Figura conſtat ex duabus Hyperbolis, Inſcriptâ & Abigenâ: Quæ eſt Species ſexageſima octava.

Species LXIX. *Fig.* 69.

At ſi deſit terminus *ey*, Figura conſtat ex duabus Inſcriptis in angulis Aſymptoton deinceps jacentibus: Quæ eſt Species ſexageſima nona.

PROP. XX. THEOR.

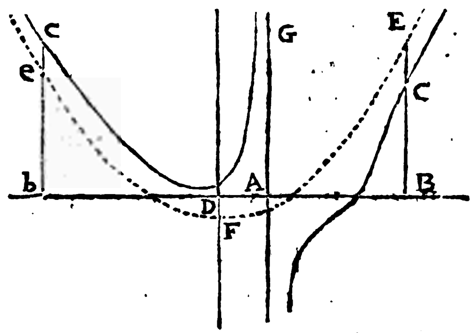
Æquatio $xy = ax^3 + bx^2 + cx + d$ *deſignat Figuram habentem duo crura Hyperbolica ad Ordinatam primam jacentia, & duo Parabolica quæ pro Aſymptoto habent Parabolam Conicam.*

SIT Abſciſſa $AB = x$, Ordinata $BC = y$. Eſt $y = ax^2 + bx + c + \frac{d}{x}$, ſit x infinite parva, & erit $y = \frac{d}{x}$, igitur (per *Coroll.* 2, *Prop.* 16.) Ordinata prima habet duo crura ad diverſas ejus partes jacentia, & in plagas oppoſitas progredientia. Augeatur jam x tam verſus dextram quam ſiniſtram in infinitum, & valor ipſius y ſemper erit affirmativus, & etiam ſine limite increſcet. Sumatur $BE = ax^2 + bx + c$, eritque (per *Prop.* 6.) locus omnium E Aſymptotos curvæ, quæ quidem eſt Parabola Conica. Q. E. D.

Coroll.

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Coroll. 1. Hæc Figura nunquam decuſſat ejus Aſymptoton *AG*. Nam tria interſections puncta abeunt in infinitum. Adeoque crura Hyperbolica & Parabolica ad eaſdem partes rectæ



AG jacentia ſemper copulantur: Et proinde hujus Figuræ Species tantum eſt unica, quæ eſt ſeptuageſima.

Coroll. 2. Parabola Conica nunquam decuſſat hanc curvam; ſecet enim, ſi fieri poſſit, & erit $EC = 0$, adeoque etiam $d = 0$, quod fieri nequit.

Coroll. 3. Sit $AD = -\frac{b}{2a}$, ordinata $DF = \frac{4ac-bb}{4a}$: Vertice *F*, Diametro *DF*, & latere recto $\frac{1}{a}$ deſcripta Parabola eſt Aſymptotos curvæ.

PROP.

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PROP. XXI. THEOR.

Æquatio $yy = ax^3 + bx^2 + cx + d$ *deſignat figuram habentem duo crura Parabolica in oppoſitas plagas errantia.* Vide fig. 70.

AUgeatur x in infinitum, & ſimul augebuntur ordinatæ y duo valores hinc inde æquales, ergo curva habet duo crura infinita. At ſi mutetur ſignum Abſciſſæ, terminus ax^3 evadet negativus: & proinde datur certus limes, ultra quem x in illas plagas non pergit. Reducendo y in ſeriem patebit crura eſſe Parabolica. Q. E. D.

Enumeratio Specierum curvæ quam deſignat æquatio $yy = ax^3 + bx^2 + cx + d$.

Species LXXI. *Fig.* 70, 71.

Si æquationis $ax^3 + bx^2 + cx + d = 0$ radices omnes ſint reales & inæquales, Parabola habet Ovalem ad verticem: Quæ eſt Species ſeptuageſima prima.

Species LXXII, LXXIII. *Fig.* 73, 74.

Si radices duæ æquentur, Figura vel prodit Nodata (*fig.* 73.) quæ eſt Species ſeptuageſima ſecunda; vel Punctata (*fig.* 74.) quæ eſt Species ſeptuageſima tertia.

Species LXXIV. *Fig.* 75.

Si tres radices æquentur, Figura erit Cuſpidata, quæ eſt Species ſeptuageſima quarta.

Eſtque

Eſtque hæc Figura Aſymptotos Parabolarum quam deſignat æquatio $yy = ax^3 + bxx + cx + d$.

Species LXXV. *Fig.* 73, 74.

Si radices duæ ſint impoſſibiles, Figura erit pura Campaniformis, Speciem conſtituens ſeptuageſimam quintam.

Species LXXVI. *Fig.* 77.

In quarto caſu æquatio erat $y = ax^3 + bx^2 cx + d$: quæ (per *Prop.* 8.) deſignat Figuram habentem crura contraria, quæ *Cubica* Parabola dici ſolet. Et ſic Species omnino ſunt ſeptuaginta ſex.

Lineæ tertii ordinis inter ſe formâ haud parum differunt, ſcilicet ut Ellipſis à circulo, vel Hyperbola æquilatera à reliquis.

Sed & aliquando vertices Figurarum quas concavas deſcripſimus, formam Cuſpidis induunt: at Cuſpides iſtiuſmodi non componuntur ex Nodis infinite parvis, nec unquam prodeunt admodum acuti.

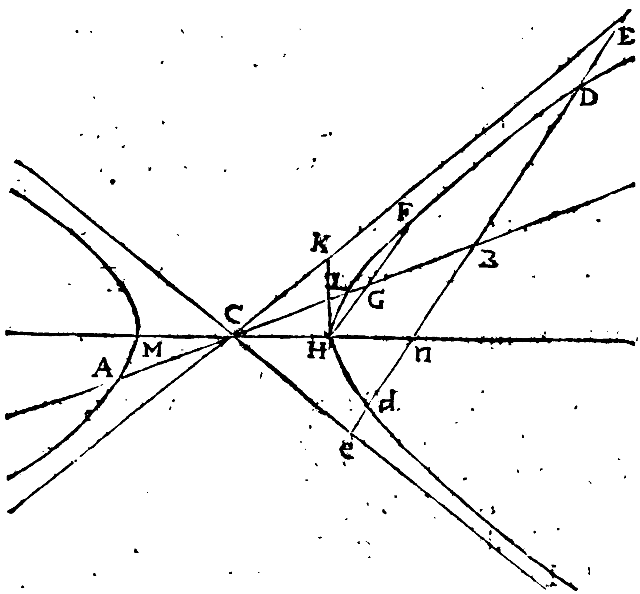
Determinatio Locorum Geometricorum.

POſtquam Species omnes Linearum alicujus ordinis enumerantur, convenit ut dignoſcatur Species quam conſtituit Linea particularis æquatione quâlibet propoſitâ denotata. Innumeræ enim æquationes, quoad formam multum inter ſe diſcrepantes, eandem Lineam deſig-

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deſignare poſſunt. Lineæ quæ excurrunt in infinitum ex Aſymptotis ſuis optime determinantur; Ovales vero ex Diametris.

Sit $y^2 + Axy + By + Cx^2 + Dx + E = 0$ æquatio univerſalis ad Coni ſectiones. Hæc æquatio (per *Prop.* 10.) facillime convertetur in hanc



$yy = Ax^2 + Bx + C$. Et Figura erit Hyperbola, Parabola aut Ellipſis prout terminus Ax^2 affirmativus eſt, nullus aut negativus.

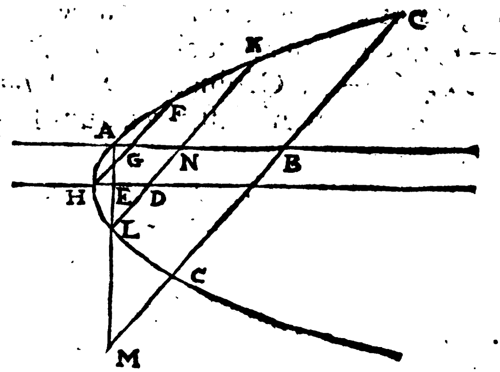
In primo caſu quando, Figura eſt Hyperbola, ſumendo Abſciſſam novam $= x + \frac{B}{2A}$, æquatio induet hanc formam $yy = Ax^2 + B$. Sit jam *CB*

Abſciſſa $=x$, Ordinata $BD=y$; & erit principium Abſciſſæ C centrum Hyperbolæ. Sume in ordinatâ BD; BE, Be hinc inde æquales x, & (per *Prop.* 6.) erunt CE, Ce duæ Aſymptoti: ducatur CHN biſecans angulum Aſymptoton, & erit CN axis. Sit H vertex Figuræ, HGF ordinatim applicata ad Diametrum CB, tranſiens per verticem. Concipiatur jam ordinata Dd motu parallelo ferri donec cum ordinatâ FH coeat; & in illo caſu erit $dN=0$, adeoque $BD=BN$. Ob datos trianguli CBN angulos, datur ratio CB ad BN, vel x ad BN. Sit ergo $x:BN::C:D$, & erit $BN=\frac{Dx}{C}$; eſt vero $BD=y=\sqrt{Ax^2+B}$. Pone ergo $(BD=)\ \sqrt{Ax^2+B}=\frac{Dx}{C}$, & invenies $x=CG=C\sqrt{\frac{B}{D^2-AC^2}}$: ſit ergo $CG:GH::C:D$: & erit H vertex, atque inde ſi erigatur normalis HK Aſymptoton ſecans in K; erunt CH, HK ſemi-axes conjugati. Datis vero CH, HK ſemi-axibus conjugatis facile deſcribitur Hyperbola. Q. E. F.

Si Figura eſt Parabola, æquatio erit $yy=Ax+B$, vel ſumendo principium Abſciſſæ in curvâ, $yy=Ax$. Sit Abſciſſa $AB=x$, ordinata BC vel $Bc=y$. Per A duc ALM perpendicularem ad Abſciſſam: ſecat ea curvam in L, ordinatim vero Cc in M. Sit $LDNK$ ordinata Abſciſſam in N ſecans. Moveatur ordinata BC motu parallelo donec coincidat cum LDK, & in illo caſu erit $Mc=0$, & proinde $BM=BC=y$.

Ob-

Ob datos trianguli *ABM* angulos datur ratio *AB* ad *BM*, vel x ad *BM*; ſit igitur $x : BM :: C : D$, & erit $BM = \frac{Dx}{C}$. Pone ergo $\frac{Dx}{C} = y =$



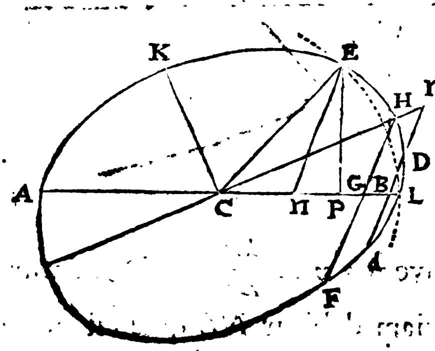
$\sqrt{Ax}$, & invenies $x = \frac{AC^2}{D^2} = AN$; datur ergo *AN*, & etiam dabitur *NL* quæ eſt ad *AN* ut *D* ad *C*. Datur ergo punctum *L*, & recta *AL* tam magnitudine quam poſitione. Biſeca *AL* in *E* & ſit *HED* normalis ad medium ejus punctum *E*, & erit *HE* axis. Sit vertex *H*, *HGF* ordinata ad Diametrum *AB* tranſiens per *H*, & in illo caſu erit $y = GH$, eſt vero *GH* data quantitas: & erit $x = \frac{GH^2}{A} = AG$. Datur ergo punctum *G*; per quod punctum ſi ducatur ordinata *GH* æqualis datæ rectæ *DN*, dabitur vertex *H*, & recta *HE*; eſt vero latus rectum

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principale æquale $\frac{AE^2}{HE}$, quod exinde dabitur. Datis jam axe *HE*, vertice *H* & latere recto $\frac{AE^2}{HE}$ deſcribi poteſt Parabola. Q. E. F.

Si Figura ſit Ellipſis, æquatio erit $yy = -Ax^2 + Bx + C$, que in hanc reducitur $yy = B - Ax^2$. Sit Abſciſſa $CB = x$, Ordinata BD vel $Bd = y$. Eritque principium Abſciſſæ centrum Figuræ.



Secet curvam in *L*, & erit $CL = \sqrt{\frac{B}{A}}$: centro *C* radio *CL* deſcribe circulum *LE* Ellipſin ſecantem in *E*: ſitque *EN* ordinata; *EP* perpendicularis in Abſciſſam; & dabitur ſpecie triangulum *NPE*. Sit ergo $NE : EP :: C : D$, id eſt $y : EP :: C : D$, unde $EP = \frac{Dy}{C}$, $PN = \frac{C}{y} \times \sqrt{C^2 - DD}$; eſt vero per naturam curvæ $CN = \sqrt{}$

$= \sqrt{\frac{B - yy}{A}}$, adeoque tota $CP = \frac{y}{C}\sqrt{C^2 - DD}$ $+ \sqrt{\frac{B - yy}{A}}$; & eſt $CE = \sqrt{\frac{B}{A}}$ adeoque æquatio $CE^2 = CP^2 + PE^2$ dabit y vel *EN*; & inde invenietur *EN* Abſciſſa correſpondens; atque adeo datur punctum *E*, & per conſequens recta *CE* poſitione. Duc rectas *KC*, *CH* biſecantes angulos *ACE*, *ECL*; & hi erunt axes, qui itaque poſitione dantur. Eorum vero magnitudo ſic determinantur: Occurrat ordinata *BD* axi in *N*, & ducatur *HGF*. Ob datum ſpecie triangulum *CGH*, datur ratio *CG* ad *GH*, ſit ergo $CH : GH$ vel $CB : BN :: C : D$, & erit $BN = \frac{Dx}{C}$. Sed cum ordinata *BD* coincidit cum *GH*, evadit *BN* æqualis $GH = y = \frac{Dx}{C}$; in æquatione ſubſtituendo illum ipſius valorem, erit $\frac{D^2x^2}{C^2} = B - Axx$, unde $x = CG = C\sqrt{\frac{B}{D^2 + AC^2}}$; adeoque dabitur ordinata *GH*; & punctum *H*, atque inde dabitur *CH* magnitudine. Et eodem modo invenire licet axem minorem.

Hæc tria Problemata ſoluta dare malui ex nudâ naturâ æquationum curvas definientium; quam per quaſdam *Apollonii* Propoſitiones more veterum Geometrarum: Hoc modo enim melius innoteſcit analogia inter Loca ſecundi ordinis & Loca tertii & ſuperiorum ordinum.

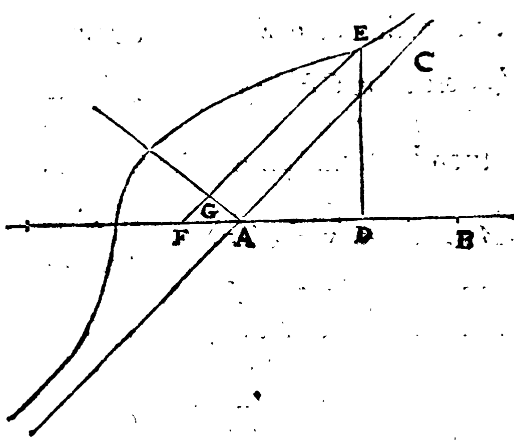
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Proprietates Sectionum Conicarum à Geometris hactenus traditæ ſufficiunt ad determinanda Loca quæ incidunt in Sectiones Coni: Conſimiles vero Linearum ſuperiorum ordinum proprietates traditas nondum habemus. Quâ ratione vero invenire licet quam Speciem conſtituit Linea tertii ordinis æquatione quâlibet deſignata, ex Propoſitione decimâ quintâ conſtare poteſt: quod per ſequentia plenius adhuc conſtabit.

Exemplum Primum.

OPorteat invenire quam ſpeciem conſtituit Linea æquatione $y^3 + x^3 = a^3$ deſignata, ubi ordinatæ inſiſtunt abſciſsâ ad angulos rectos.



Mutetur ſignum Abſciſſæ & æquatio evadet $y^3 = x^3 + a^3$; & inde (per Theorema *Neutoni*) $y = x + \frac{aaa}{3x^2} + \&c.$ Sit Abſciſſa *AB*, duc rectam

AC

AC per initium Abſciſſæ, quæ cum Abſciſsâ angulum ſemirectum contineat, & (per *Prop.* 6.) erit *AC* Aſymptotos. Et quoniam unica tantum iſtiuſmodi ſeries obtineri poteſt, curva habet niſi duo crura infinita ad eandem Aſymptoton jacentia, & (per *Coroll.* 2. *Prop.* 6.) hæc crura jacent ad eaſdem Aſymptoti partes; & inde (per *Coroll.* 7. *Prop.* 16.) curva habet diametrum ad ordinatas duarum dimenſionum. Sit *E* punctum quodlibet in curvâ, Ordinata $ED = y$, Abſciſſa $AD = x$; Ordinata nova Aſymptoto parallela $FE = v$, Abſciſſa correſpondens $AF = z$; erit $y = v\sqrt{\frac{1}{2}}$, $x = v\sqrt{\frac{1}{2}} - z$. In æquatione $y^3 = x^3 + a^3$, hoſce valores ſubſtitue, & proveniet $\frac{1}{2}v^3\sqrt{\frac{1}{2}} = \frac{1}{2}v^3\sqrt{\frac{1}{2}} - \frac{3}{2}zv^2 + 3z^2v\sqrt{\frac{1}{2}} - z^3 + a^3$, vel $3zv^2 - 6z^2v\sqrt{\frac{1}{2}} = 2a^3 - 2z^3$. Unde (per *Prop.* 10.) $z\sqrt{\frac{1}{2}}$ eſt ordinata Lineæ biſecantis ordinatas v ad curvam terminatas: Duc igitur rectam *AG* per principium Abſciſſæ Aſymptoton ſecantem ad rectos angulos; & illa erit recta biſecans ordinatas. Occurrat ea ordinatæ *EF* in *G*; & ſit Abſciſſa nova $AG = x$, ordinata Correſpondens $EG = y$. Erit $z = x\sqrt{2}$, $v = y + x$; in æquatione $3zv^2 - 6z^2v\sqrt{\frac{1}{2}} = 2a^3 - 2z^2$ hoſce valores pro z, v ſubſtitue, & orietur $3xy^2\sqrt{2} = 2a^3 - x^3\sqrt{2}$, vel ponendo $a = c\sqrt{2}$, $3xy^2\sqrt{2} = 4c^3\sqrt{2} - x^3\sqrt{2}$, & dividendo per $\sqrt{2}$, $3xy^2 = 4c^3 - x^3$: Cumque æquatio hæc ſit ejuſdem formæ cum hâc $xy^2 - ey = ax^3 + bx^2 + cx + d$, ubi deeſt terminus ey, & ax^3 eſt negativus, & præterea æquationis $ax^3 + bx^2 + cx + d$ radices duæ ſunt impoſſibiles, Figura conſtituit Speciem quinquageſimam nonam. Q. E. I.

Exem-

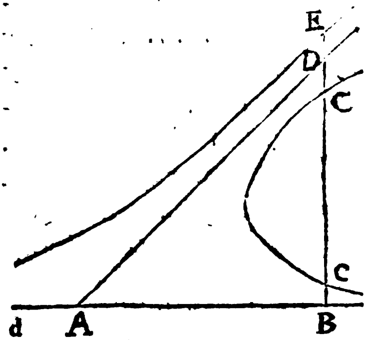
Exemplum Secundum.

OPorteat invenire quam ſpeciem conſtituit Linea æquatione $y^3 - 2xy^2 + x^2y - a^3 = 0$ deſignata, ubi ordinatæ inſiſtunt abſciſſæ ad angulos rectos: hujus æquationis tres radices ſeu valores ipſius y reductæ in ſeries eo citius convergentes quo major eſt abſciſſa x ſunt

$$x + \frac{\sqrt{ax}}{aa} - \frac{a^3}{2x^2} \text{ \&c.}$$

$$x - \frac{aa}{\sqrt{ax}} - \frac{a^3}{2x^2} \text{ \&c.}$$

$$\frac{a^3}{x^2} + \frac{2a^6}{x^5} + \frac{7a^9}{x^8} + \text{\&c.}$$



Sit Abſciſſa $AB = x$; Ordinata rectangula $BcCy$: in illâ ordinatâ ſit $BD = AB$, junge AD, & (per *Prop.* 6.) erit illa Aſymptotos, quæ pro duabus Aſymptotos coeuntibus habenda eſt; ſiquidem duo termini priorum ſerierum coincidunt: Unde hæc Linea vel conſtituit ſpeciem ſexageſimam octavam vel ſexageſimam nonam, nam hæ duæ ſpecies ſolæ ſunt Linearum tertii ordinis quæ habent Aſymptotos duas coincidentes. Series tertia $\frac{a^3}{x^3} + \frac{2a^6}{x^5} + \frac{7a^9}{x^8} +$ &c. indicat Abſciſſam eſſe Aſymptoton habentem duo crura ad oppoſitas plagas protenſa & ad eaſdem ejus partes jacentia, quoniam mutando ſignum abſciſſæ terminus $\frac{a^3}{x^2}$ manet affirmativus. Et quoniam in extremitate Aſymptoti AD ad diſtantiam infinitam ſitâ jacet punctum curvæ duplex; Aſymptotos illa habet crura ſua ad diverſas ejus partes jacentia, & in plagas eaſdem protenſa. Unde facile videre eſt hanc curvam conſtituere Speciem ſexageſimam nonam: Habetque Aſymptoton AD pro Diametro ad ordinatas duarum dimenſionum, quæ parallelæ ſunt Aſymptoto AB. Q.E.I.

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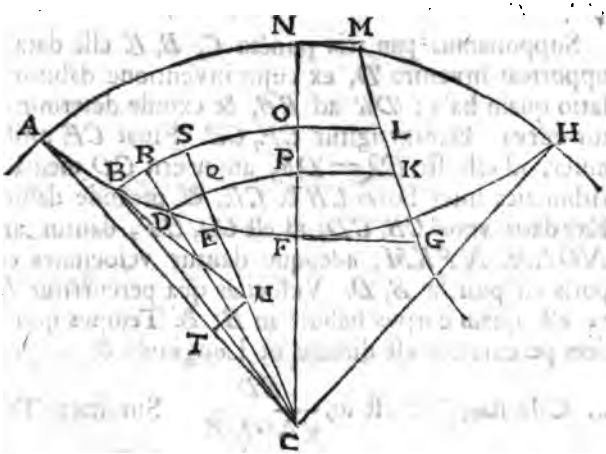
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Inventio Lineæ celerrimi deſcenſus in quacunque hypotheſi gravitatis.

PROBLEMA.

Invenire Lineam celerrimi deſcenſûs, datâ lege vis Centripetæ.

SIT *C* centrum virium, *ADFH* linea quæſita, *A* punctum de quo corpus cadere debet, *B*, *D*, *E* tria puncta, quorum diſtantiæ ſunt quam minimæ; junge *CB*, *CD*, *CE*: Centro *C* & radiis *CB*, *CD*, deſcribe circulos *BO*, *DP*, quorum hic ſecet *CN*



(axem curvæ) & *CE* in *P*, *Q* reſpective: ille vero rectas *CD*, *CE* in *R*, *S*; axem vero in *O*.

* A Sit

Sit *MLKG* linea cujus Ordinatæ *NM*, *OL*, *PK*, *FG*, inſiſtentes Abſciſsâ *GN* normaliter proportionales ſunt vi centripetæ in punctis *N*, *O*, *P*, *F*, reſpectivè. Cadat jam Corpus à puncto *N*, versùs centrum vi ſolâ centripetâ agitatum, & (per Prop. 39. Lib. 1. *Princip. Newtoni*) ejus velocitas in puncto quovis *O* erit ut areæ *NOLM* latus quadratum.

Jaceant puncta *A*, *N* in circumferentiâ Circuli cujus centrum *C*, & velocitates corporum in curvâ *AB* & recta *NC* motorum in omnibus æqualibus à centro *C* diſtantiis erunt æquales. Nam (per Prop. 40. Lib. 1. *Newtoni*) ſi eorum velocitates in aliquâ æquali altitudine ſint æquales, in omni æquali altitudine æquales erunt: at corporum iſtorum velocitates in punctis *A*, *N*, erant æquales, quippe nullæ; ergo & in omnibus diſtantiis æqualibus æquales erunt. Igitur velocitas corporis in curva *ABF* moti, per Areæ curvi lineæ *MLG* latus quadratum rite repreſentabitur: ſcilicet velocitas in *B* eſt ut $\sqrt{NOLM}$, velocitas in *D* eſt ut $\sqrt{NPKM}$.

Supponamus jam tria puncta *C*, *B*, *E* eſſe data, & opporteat invenire *D*, ex cujus inventione dabitur relatio quam habet *DR* ad *RB*, & exinde determinabitur curva. Dantur igitur *CE*, *CB*. Fluat *CE* uniformiter, id eſt, ſit $EQ = DR$, atque erit *CD* media Arithmetica inter Datas *CB* & *CE*, & proinde dabitur. Ex datis vero *CB*, *CD*, id eſt *CO*, *CP*, dantur areæ *NOLM*, *NPKM*; adeoque dantur velocitates corporis ad puncta *B*, *D*. Velocitas qua percurritur *BD* ea eſt quam corpus habuit in *B*; & Tempus quo eadem percurritur eſt directe ut Longitudo & reciproce ut Celeritas, hoc eſt ut $\frac{BD}{\sqrt{NOLM}}$. Similiter Tempus quo percurritur *DE* eſt ut $\frac{DE}{\sqrt{NPKM}}$; & inde

Tempus

Tempus quo percurritur BE eſt ut $\frac{BD}{\sqrt{NOLM}} + \frac{DE}{\sqrt{NPKM}}$: ſed quia tempus quo tota curva percurritur ſupponitur breviſſimum, erit tempus per quamlibet ejus partem etiam breviſſimum. Et proinde fluxio quantitatis huic tempori proportionalis æqualis nihil.

Hiſce præmiſſis, ſit $CD = CP = x$; $DR = PO = \dot{x}$, $BR = y$, $DQ = z$, erit $BD = \sqrt{\dot{x}^2 + \dot{y}^2}$, $DE = \sqrt{\dot{x}^2 + \dot{z}^2}$: hiſce valoribus pro BD, DE ſubſtitutis, erit $\sqrt{\frac{\dot{x}^2 + \dot{y}^2}{NOLM}} + \sqrt{\frac{\dot{x}^2 + \dot{z}^2}{NPKM}}$ quantitas tempori proportionalis, adeoque ejus fluxio $= 0$, hoc eſt,

$$\frac{\dot{y}\ddot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}\,\sqrt{NOLM}} + \frac{\dot{z}\ddot{z}}{\sqrt{\dot{x}^2 + \dot{z}^2}\,\sqrt{NPKM}} = 0;$$

nam $\dot{x}$ & areæ $NOLM$, $NPKM$ ſunt quantitates non fluentes. Ob data tria puncta C, B, E; datur BS (normalis à vertice Trianguli CBE in baſin CE Demiſſa) $= BR + RS = BR + DQ = y + z$, adeoque $y + z$ eſt data quantitas, & ejus fluxio $\dot{y} + \dot{z} = 0$, vel $\ddot{z} = -\ddot{y}$. In æquatione $\frac{\dot{y}\ddot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}\,\sqrt{NOLM}} = \frac{-\dot{z}\ddot{z}}{\sqrt{\dot{x}^2 + \dot{z}^2}\,\sqrt{NPKM}}$ pro $\ddot{z}$ pone ejus valorem

$+\ddot{y}$ & divide æquationem per $\ddot{y}$ atque erit

$$\frac{\dot{z}}{\sqrt{\dot{x}^2+\dot{z}^2}\sqrt{NPHM}} = \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}\sqrt{NOLM}}.$$

Cumque hoc univerſaliter in omnibus curvæ punctis obtineat, patet eſſe $\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}\sqrt{NOLM}}$ datam quantitatem, vel $\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}\sqrt{NOLM}} = \frac{1}{A}$ (ubi A eſt data quantitas) & $A\dot{y} = \sqrt{\dot{x}^2+\dot{y}^2}\sqrt{NOLM}$: quæ æquatio determinat curvam. Q. E. I.

Coroll. 1. Eſt $A^2 = NFGM$. nam cum recta CB vel x coincidit cum CF, ea eſt normalis in curvam & $\dot{x} = 0$, atque area $NOLM$ evadit $=$ areæ $NFGM$. In æquatione igitur $A\dot{y} = \sqrt{\dot{x}^2+\dot{y}^2}\sqrt{NOLM}$ evaneſcat $\dot{x}$ & ſit $NOLM = NFGM$, atque erit $A\dot{y} = \sqrt{\dot{y}^2}\sqrt{NFGM}$, id eſt, $A\dot{y} = \dot{y}\sqrt{NFGM}$ & dividendo per $\dot{y}$, $A = \sqrt{NFGM}$. Q. E. D.

Coroll. 2. $BD : BR$ ut velocitas maxima quam corpus in motu per curvam AF acquirit ad ejus velocitatem in puncto B. Nam eſt $A\dot{y} = \sqrt{\dot{x}^2+\dot{y}^2}\sqrt{NOLM}$, vel $\sqrt{NFGM}\times\dot{y} = \sqrt{\dot{x}^2+\dot{y}^2}\sqrt{NOLM}$, adeoq; $BD : BR :: \sqrt{NFGM} : \sqrt{NOLM}$; hoc eſt, ut velocitas maxima ad velocitatem in puncto B.

Coroll.

Coroll. 3. *BR* : *RD* : : √*NOLM* : √*FOLG*.

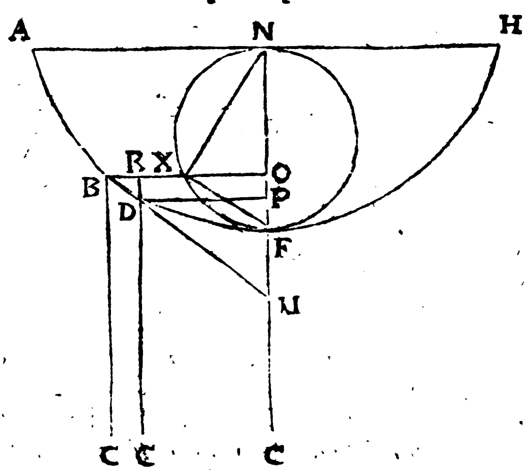
Coroll. 4. Si ſint *A*, *H*, puncta curvæ altiſſima, & ducantur *CA*, *CH*, hæ tangent curvam in punctis *A*, *H*.

Coroll. 5. Igitur in nulla hypotheſi gravitatis Recta linea eſt via celerrimi deſcenſus; præterquam ubi corpus deſcendens directe tendit ad centrum.

Coroll. 6. Sit *BT* longitudinis cujuſvis, & *BT* : *TU* : : √*FOLG* : √*NOLM* : ſit vero *TU* in *TB* normalis, junge *BU*, & illa Tanget curvam in puncto *B*. Nam *DR* : *BR* : √*FOLG* : √*NOLM*, ergo *DR* : *BR* : : *BT* : *TU*; & proinde *BU* tangit curvam ad punctum B. Q. E. D.

Proprietates hucuſque traditæ ſunt generales, Lineæ celerrimi deſcenſûs in omni hypotheſi gravitatis univerſaliter competentes. Deſcendamus jam ad caſus particulares.

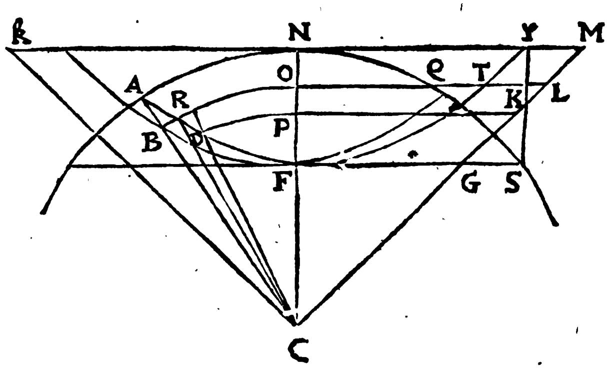
Exemplum primum.



Upponamus vim centripetam eſſe uniformem & agere in parallelis: quo in caſu punctum *C* abit in infinitum,

infinitum, exiſtentibus *CF*, *CD*, *CB* parallelis. Peripheriæ *AN*, *BO*, *DP* & curva *MG* migrant in rectas, & area *NOLM* in rectangulum, atque area *NOLM* ad aream *NFGM* ut *NO* ad *NF*, & inde $BD : BR :: \sqrt{NF} : \sqrt{NO}$. Supra diametrum *NF* deſcribe circulum *NXF* ſecantem ordinatam *BO* in *X*, junge *NX*, *FX*, & *BU* ſit tangens ad *B*. Propter ſimilia Triangula *NXF*, *NXO*, *FXO*. $NF : NX :: NX : NO$. ergo $\sqrt{NF} : \sqrt{NO} :: NF : NX$. & $NF : NX :: FX :: XO$. unde $DB : BR :: FX : XO$. duc tangentem *BU*, & erit $BD : BR :: BU : BO$, adeoque $BU : BO :: XF : XO$. ergo tangens *BU* ſemper parallela eſt Chordæ *XF* quæ eſt notiſſima proprietas Cycloidis vulgaris.

Exemplum ſecundum.



SIT vis centripeta ut diſtantia à centro, & curva *MLG*, migrabit in rectam tranſeuntem per centrum *C*. ſint anguli *KCN*, *kCN* ſemirecti. $BR : DR :: \sqrt{NOLM} : \sqrt{FOLG}$. Jam ſit $CF = a$, $CO = x$, $CN = r$, & erit area $NOLM = \frac{1}{2} r^2 - \frac{1}{2} x^2$, $FOLG = \frac{1}{2} x^2 - \frac{1}{2} a^2$, unde $x : y :: \sqrt{x^2 - a^2}\ \sqrt{r^2 - x^2}$. Deſcribatur

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ſcribatur Hyperbola *FTr*, vertice *F* & Aſſymtotis *CK*, *Ck*. Occurrat *OL* Hyperbolæ in *T*, & circulo in *Q*; atque erit per hujus Hyperbolæ & circuli naturam $OT = \sqrt{x^2 - a^2}$, $OQ = \sqrt{rr - xx}$. ergo $DR : BR :: OT : OQ$.

Hujus Curvæ Rectificatio, per quadrat. Hyperb.

Patet $BD : DR :: FS : OT$. hoc eſt, incrementum curvæ ad incrementum Axis *OP* ut data recta *FS* ad Hyperbolæ ordinatam reſpectivam *OT*. Erit igitur componendo, ut omnes *BD* ad omnes *OP* ita totidem *FS* ad totidem ordinatas Hyperbolæ. Hoc eſt ut curva *AF* ad axem ejus *NF*, ita rectangulum *NFSr* ad aream Hyperbolicam *NFTr*. Etenim area illa *NFTr* eſt ſumma ordinatarum *OT*.

Erit etiam ut *AB* ad *NO* ita $Nr \times OF$ ad aream *OFT*.

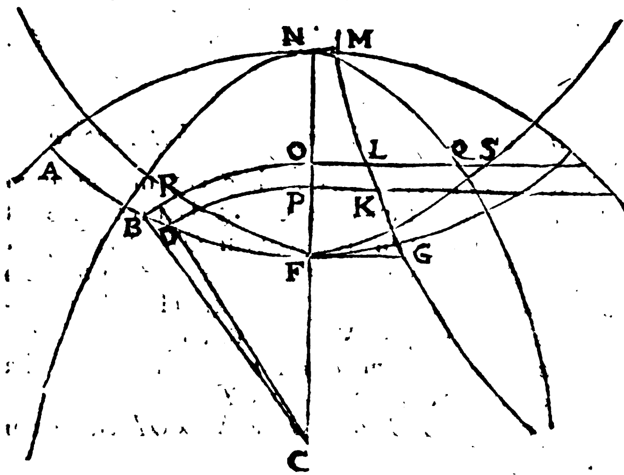
Hujus Curvæ Quadratura per quad. Hyp. & Circ.

Fluxio areæ, ſcil. triangulum $CBD = \frac{1}{2} CB \times BR = \frac{1}{2} x \times \dot{y} = \frac{1}{2} x \dot{x} \sqrt{\frac{r^2 - x^2}{x^2 - a^2}}$. Unde $CBD : \frac{1}{2} x \dot{x} :: \sqrt{r^2 - x^2} : \sqrt{x^2 - a^2}$, Eſt vero *CBD* fluxio areæ *CFB*, & $\frac{1}{2} x \dot{x}$, fluxio quantitatis $\frac{1}{4} x^2$, ergo ut area *CFB* ad $\frac{1}{4} xx$ ita omnes $\sqrt{r^2 - x^2}$ ad omnes $\sqrt{x^2 - a^2}$; hoc eſt $CFB : \frac{1}{4} xx ::$ area *FOQS* : aream *FOT*. Coincidat jam *CB* cum *CF* & evaneſcet area *CFB*, at evadet $\frac{1}{4} xx = \frac{1}{4} CF^2 = \frac{1}{4} aa$. Igitur ſtatim apparet $\frac{1}{4} xx$ fluentem quantitatis $\frac{1}{2} x \dot{x}$ minui debere quantitate $\frac{1}{4} aa$, & proinde erit vere $CFB : \frac{1}{4} xx - \frac{1}{4} aa ::$ *FOQS* : *FOT*. & Area $ABC : \frac{1}{4} xx - \frac{1}{4} aa ::$ *NOQ* : *NOTr*. atque area tota $CFA : \frac{1}{4} rr - \frac{1}{4} aa ::$ *NSF* : *NF*.

Exem-

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Exemplum tertium.



SIT vis centripeta reciproce ut quadratum diſtantiæ à centro; & erit ordinata $OL = \frac{1}{xx}$, adeoq; per notas quadrandi methodos $FOLG = \frac{1}{a} - \frac{1}{x}$, $NOLM = \frac{1}{x} - \frac{1}{r}$; & inde $BR^2 : RD^2 :: \frac{1}{x} - \frac{1}{r} : \frac{1}{a} - \frac{1}{x} :: \dot{y}^2 : \dot{x}^2$, vel, quod idem eſt, $\dot{x}^2 : \dot{y}^2 :: rx - ra : ra - ax$, & $\dot{x} : \dot{y} : \sqrt{rx - ra} : \sqrt{ra - ax}$. Igitur latere recto r, axe FN, vertice F deſcribe Parabolam FS; item latere recto a axe NF, vertice N deſcribe Parabolam NQ: occurrat ordinata OL Parabolis hiſce in Q, S; & erit $\dot{x} : \dot{y}$ vel $DR : BR :: OS : OQ$. Nam eſt $OS = \sqrt{rx - ra}$, & $OQ = \sqrt{ra - ax}$.

Hujus

Hujus curvæ rectificatio.

Quoniam $x^2 : y^2 :: rx - ra : ra - ax$, erit $x^2 + y^2 : x^2 :: rx - ax : rx - ra$; & inde $\sqrt{\dot x^2 + \dot y^2} : \dot x :: \sqrt{rx - ax} : \sqrt{rx - ra} :: BD : DR$, hoc eſt, ut fluxio curvæ ad fluxionem Abſciſſæ: & componendo, erit ut ſumma omnium $\sqrt{\dot x^2 + \dot y^2}$ ad omnes $\dot x$, ita omnes $\sqrt{rx - ax}$ ad tot $\sqrt{rx - ra}$, hoc eſt, $BF : FQ :: x\sqrt{rx - ax} - a\sqrt{ar - aa} : \overline{x - a}\sqrt{rx - ra}$ (nam $x\sqrt{rx - ax}$ fluens fluxionis $\dot x\sqrt{rx - ax}$ minuenda eſt quantitate $a\sqrt{ar - aa}$.)

Curva hæc perfecte quadrabilis eſt, nam $\dot x : \dot y :: \sqrt{rx - ra} : \sqrt{ra - ax}$, ergo $x\dot x : x\dot y :: \sqrt{rx - ra} : \sqrt{ra - ax}$; hoc eſt, $x\dot x : 2\dot{CBD}$ vel $\frac{1}{2} x\dot x : \dot{CBD} :: \sqrt{rx - ra} : \sqrt{ra - ax}$. Et ſumendo ſummas omnium, erit $\frac{1}{4} xx : CFB :: \overline{x - a}\sqrt{rx - ra} : \overline{r - a}\sqrt{ra - aa} - \overline{r - x}\sqrt{ra - ax}$. Et erit tota area CFA (comprehenſa rectis CF, CA & curva AF) ad $\frac{1}{4} CN^2$ ut $\sqrt{CN^2 - CN \times CF}$ ad $\sqrt{CN \times CF - CF^2}$.

Hæ ſunt tres inſigniores Hypotheſes Gravitatis, etenim ſi praxin ſpectemus, tuto ſupponi poteſt uniformis, & agere in parallelis, at in rigore geometrico, hypotheſes duæ ultimæ in rerum natura locum vere habent. Nam in terræ cavernis gravitas eſt ut diſtantiæ corporis à centro Terræ, & in Turrium vel montium cacuminibus ea eſt reciproce in duplicata ratione diſtantiæ, ut conſtat ex principiis *Newtoni*.

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Data Lineæ Celeritatum deſcenſus, invenire legem vis Centripetæ.

Æquatio definiens curvam, AFH (fig. 1.) eſt $A\dot{y}=\sqrt{\dot{x}^2+\dot{y}^2}\,\sqrt{NOLM}$. ubi $CO=CB=x$, $DR=OP=\dot{x}$, $BR=\dot{y}$, $NFGM=A^2$. Jam area indeterminata $NOLM$ dicatur z^2, & erit $A\dot{y}=z\sqrt{\dot{x}^2+\dot{y}^2}$, & inde $zz=\frac{A^2\dot{y}^2}{\dot{x}^2+\dot{y}^2}=NOLM$. hujus æquationis capiatur fluxio, & erit $z\dot{z}=\frac{A^2\dot{x}^2\dot{y}\ddot{y}}{\overline{\dot{x}^2+\dot{y}^2}|^2}=\frac{1}{2}POLK=\frac{1}{2}PO\times LO=\frac{1}{2}\dot{x}\times OL$. Unde dividendo per $\dot{x}$ habebimus Ordinatam $OL=\frac{2A^2\dot{x}\dot{y}\ddot{y}}{\overline{\dot{x}^2+\dot{y}^2}|^2}$ vel $OL=\frac{A^2\dot{x}\dot{y}\ddot{y}}{\overline{\dot{x}^2+\dot{y}^2}|^2}$; ex data vero ordinata OL ad Abſciſſam x pertinente, dabitur curvæ MLG. Q. E. I.

Methodus diſponendi quotcunque Sphæras in Fornicem. Et inde Demonſtratur Proprietas præcipua Curvæ Catenariæ.

PROBLEMA.

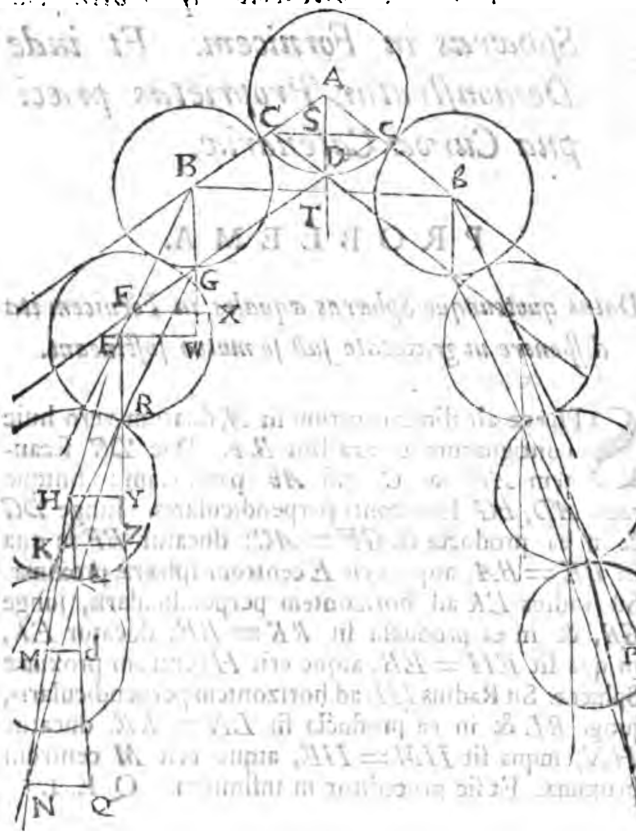
Datas quotcunque Sphæras æquales in Fornicem ita diſponere ut gravitate ſuâ ſe mutuo ſuſtineant.

SPhæræ altiſſimæ centrum ſit *A*, duarum vero huic contiguarum centra ſint *B*, *b*. Duc *DC* ſecantem *AB* in *C* ipſi *Ab* parallelam: ſintque radii *AD*, *BG* Horizonti perpendiculares. Junge *DG* & in ea producta ſit *GF* = *AC*: ducatur *BF* in qua ſit *BE* = *BA*; atque erit *E* centrum ſphæræ proximæ. Sit radius *ER* ad horizontem perpendicularis, junge *GR*, & in ea producta ſit *RK* = *BF*; ducatur *EK*, in qua ſit *EH* = *EB*; atque erit *H* centrum proximæ Sphæræ. Sit Radius *HL* ad horizontem perpendicularis, junge *RL* & in ea producta ſit *LN* = *KE*. ducatur *HN*, inqua ſit *HM* = *HE*, atque erit *M* centrum proximæ. Et ſic proceditur in infinitum. Q. E. I.

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Demonſtratio.

Omnes hæ Sphæræ triplici potentia urgentur: & conſtat ex *Mechanica*, quod tres potentiæ in æquilibrio conſiſtentes eam ad ſe invicem rationem habent, quam [illegible]



tres rectæ potentiarum directionibus reſpective parallelæ & ad earum interſectiones mutuas terminatæ. Sphæra *AT* urgetur gravitate ab *A* verſus *D* tendente, & actione

actione ſphærarum contiguarum in rectis *AB*, *Ab*. Adeoque ſi radius *AD* exponat gravitatem Sphæræ abſolutam, *AC* exponet vim qua Sphæra *AD* urget Sphæram *BG* à *B* verſus *E*. Item *BF* exponet vim qua Sphæra *ER* urget Sphæram *HL*, *EK* vim qua Sphæra *HL* urget Sphæram ſequentem. Et ſic in infinitum. Vis *BF* reſolvitur in vires *BG*, *GF*: id eſt, *BG*, *AC*: & iiſdem æquipollet. Eſt vero *BG* vis gravitatis globi & *AC* vis qua Sphæra *AD* urget Sphæram *BG*: & hæ duæ vires componunt omne pondus ſuſtinendum à Sphæra *ER*; ſcilicet unaquæque Sphæra ſuſtinere debet omne pondus quod ſuſtinet proxime ſuperior Sphæra una cum ipſius gravitate abſoluta. Et vis *EK* æquipollet viribus *ER*, *RK* vel *ER*, *BF* id eſt vi gravitatis Sphæræ *ER* & omni pondere quam eadem ſuſtinet. Atque ſic pergendo in infinitum, videbis ubique Sphærarum ſitum ita eſſe comparatum, ut earum quælibet ſuſtinere poteſt omne pondus quod ſuſtinet Sphæra immediate ſuperior, una cum ipſius gravitate abſoluta: adeoque hæ Sphæræ erunt in æquilibrio, & vi gravitatis ſeſe ſuſtinebunt.

Theorema.

Sint *M* & *H* centra duarum Sphærarum contiguarum; duc *Hd*, *Md*; hanc horizonti parallelam, illam vero eidem normalem. Dico ſemper eſſe *Md* ad *dH* ut data quædam recta ad ſummam diametrorum Sphærarum omnium quam ſuſtinet Sphæra cujus centrum *M*.

A centris Sphærarum *B*, *E*, *H*, *M*, &c. in radios productos (ſi opus eſt,) *AD*, *BG*, *ER*, *HL*, &c. demitte normales *BT*, *EW*, *HY*, *MD*, &c. In eaſdem radios ſint etiam perpendiculares *CS*, *FX*, *KZ*, *NQ*, &c. Patet ex Conſtructione eſſe $CS = FX = KZ = NQ$. adeoque eſt *NQ* data recta, quæque ex angulo *BAD* aſſumpto ad libitum determinatur. Ex conſtructione

ctione etiam patet eſſe $AS = \frac{1}{2} AD = GX$, $RT = \frac{3}{2} AD$. & $EZ = \frac{5}{2} AD$, $HQ = \frac{7}{2} AD$, &c. Et eſt Md ad dH ut NQ ad QH, hoc eſt, ut data recta ad $\frac{7}{2} AD$. vel ut $2 NQ$ ad $7AD$, eſt vero $7AD$ ($= AB + BE + EH + HM$) ſumma diametrorum omnium Sphærarum quas ſuſtinet Sphæra, cujus centrum M. Conſtat ergo Theorema, Q. E. D.

Supponamus filum tenue tranſire per centra omnium Sphærarum, cujus extremitates fixæ ſunt in punctis M, P; & tum Sphæras deorſum converti; atque illæ libere pendentes ſitum priorem inter ſe retinebunt. Nam potentiarum ſolummodo directiones, non magnitudines mutantur.

Augeatur numerus Sphærarum & minuantur earum Diametri in infinitum, & filum illud evadet Curva *Catenaria*; & Md erit incrementum Ordinatæ, dH incrementum Abſciſſæ: & ſumma diametrorum Sphærarum quas ſuſtinet Sphæra cujus centrum M, erit longitudo curvæ inter verticem A & punctum M intercepta, & quantitas data NQ evadet radius Curvaturæ in vertice.

Unde Catenæ proprietas hæc eſt; ut incrementum Ordinatæ ad incrementum Abſciſſæ, ita duplus Radius curvaturæ in vertice ad longitudinem curvæ inter verticem & ordinatam illam interceptæ. Q. E. D.

Et ſimiliter invenies Figuram Fornicis vel Catenæ in quacunque hypotheſi gravitatis, quamvis Sphæræ non ſunt æquales, vel etiam ſi eſſent Sphæroides.

Solutio

Solutio Problematis à Leibnitio nuper propoſiti.

PROBLEMA.

Invenire Lineam quæ ad angulos rectos ſecabit omnes Hyperbolas Conicas iiſdem verticibus & circa eundem axem deſcriptas.

SIT recta *PGDB* axis, *F* centrum & *G*, *D* vertices Hyperbolarum. *A* punctum in axe quodlibet per quod tranſire ſupponitur curva quæſita *ACZQ*; quam tangat *CH* in *C*, cui normalis ſit *CE*; quæ propterea tanget Hyperbolam in *C* quæ axem habet *GD*, vertices *G*, *D* & tranſit per punctum *C*. Unde (per naturam Hyperbolæ) eſt *BF* ad *FD* ſicut *FD* ad *FE*.

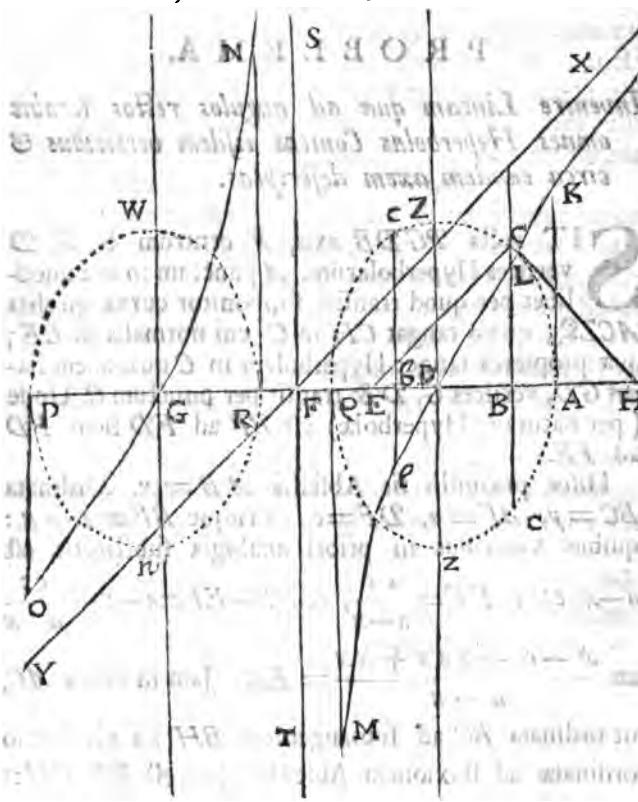
Hiſce præmiſſis, ſit Abſciſſa $AB = x$, Ordinata $BC = y$, $AF = a$, $DF = c$. Eritque $BF = a - x$: quibus valoribus in priori analogia ſubſtitutis, eſt

$$a - x : c :: c : EF = \frac{cc}{a - x}, \text{ \& } BF - EF = a - x - \frac{cc}{a - x}$$

$$= \frac{a^2 - c^2 - 2ax + xx}{a - x} = EB.$$ Jam in curva *AC*, ut ordinata *BC* ad ſubtangentem *BH* ita eſt fluxio ordinatæ ad fluxionem Abſciſſæ: hoc eſt $BC : BH :: \dot{y} : \dot{x}$: propter vero ſimilia triangula *EBC*, *CBH*, eſt $EB : BC :: BC : BH$; unde ex æquo $EB : BC :: \dot{y} : \dot{x}$. ubi ſi pro *EB* & *BC* ſubſtituantur earum valores,

proveniet

proveniet $\frac{a^2 - c^2 - 2ax + x^2}{a - x} : y :: \dot{y} : \dot{x}$. Ergo $y\dot{y} = \dot{x} \times \frac{a^2 - c^2 - 2ax + x^2}{a - x}$ eſt æquatio curvam quæſitam AC definiens. Q. E. I.



Patet hanc æquationem à fluxionibus liberari non poſſe, nam quantitas $\dot{x} \times \frac{a^2 - c^2 - 2ax + x^2}{a - x}$ eſt fluxio

xio

xio areæ hyperbolæ, cujus ordinata eſt $\frac{a^2 - c^2 - 2ax + x^2}{a - x}$ pertinens ad abſciſſam x. Hæc autem Hyperbola Aſymptotos habet SFT, & XFY, quarum illa eſt axi GD normalis, hæc vero cum illa continet angulos SFX, YFT ſemirectos; & jacet Hyperbola in angulis illis deinceps, per Hyperbolarum vertices G, D tranſiens.

Hac Hyperbola ſemel deſcripta; ut in Schemate, Curvam ſic conſtruo. Ducatur ordinata prima AK ſecans Hyperbolam in K, & occurrat ordinata quævis alia BC eidem in L; atque erit $\dot{x} \times \frac{a^2 - c^2 - 2ax + x^2}{a - x}$ vel $y\dot{y}$ fluxio areæ Hyperbolicæ $AKLB$; unde regrediendo ad fluentes, erit $\frac{1}{2}yy =$ areæ $AKLB$, & $yy = 2\,AKLB$, atque $y = \pm\sqrt{2\,AKLB} = BC$ vel Bc, qui duo valores propter contraria ſigna ad partes Abſciſſæ contrarias jacent.

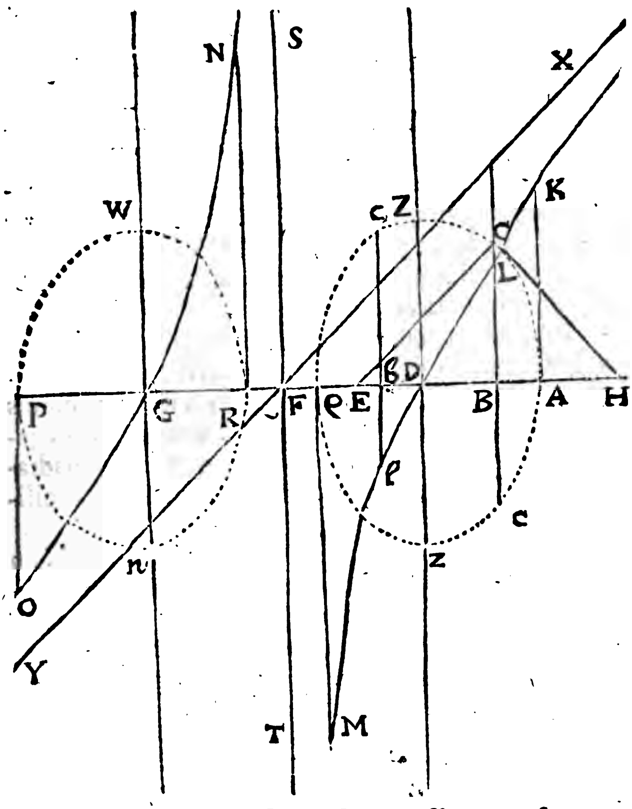
1. Patet hanc curvam ad A incipere, ubi incipit area AL, nec ultra punctum A verſus H extendi. Nam ſi area $AKLB$ pro affirmativa habeatur, omnis area alia quæ ad alias Abſciſſæ vel Ordinatæ partes jacet erit negativa; & area negativa, quæ latus quadratum non admittit, demonſtrat ordinatæ impoſſibilitatem.

2. Ordinata maxima tranſit per punctum D. Nam area $AKLB$; (cujus radici quadratæ æqualis eſt ordinata) continuo creſcit dum progreditur punctum B ab A verſus D. poſtquam vero ordinata BL ultra punctum D in ſitum bl pervenit, minuenda eſt area AKD area blD, ob arearum plagas contrarias, & tunc erit $bc = \sqrt{2\,AKD - 2\,blD}$.

3. Et Ordinata bc progrediendo à D ad Q continuo minuitur donec tandem ad Q evaneſcit; tunc exiſtente area affirmativa AKD æquali areæ negativæ QMD.

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4. Si ſit $FR = FQ$ ad partes contrarias ſita, ordinata imaginaria erit inter puncta Q & R, ob negativam aream prævalentem, & ad R iterum incipiet eſſe realis. Etenim area, cujus lateri quadrato æqualis eſt ordinata, manebit negativa donec area affirmativa infinita *FRNS* evadet æqualis areæ negativæ infinitæ *FQMT*.



5. Atque inter puncta R & G ordinata eodem modo increſcet, quo prius decrevit inter puncta D, Q; & inter puncta G, P, ubi eſt $FA = FP$, ordinata rurſus

rurſus continuo decreſcet, atque tandem ad punctum *P* evaneſcet.

6. Ex his omnibus conjunctim conſtat Lineam de qua quærebatur, fore Curvam Irrationalem quarti ordinis (quam ſcilicet recta in quatuor tantum punctis ſecare poteſt) conſtantem ex duabus Ovalibus æqualibus, ſimilibus, & ſimiliter poſitis, quæ habent punctum duplex in plaga Ordinatarum ad diſtantiam infinitam.

Radius Curvaturæ ad punctum *A* æqualis eſt *AK* ordinatæ Hyperbolæ per punctum *A* tranſeuntis. Concipe enim Abſciſſam *AB* eſſe infinite parvam, & radius curvaturæ ad punctum *A* æqualis erit $\frac{BC^2}{2AB} = \frac{y^2}{2x}$: eſt vero in prima ſua magnitudine $yy = 2AB \times AK \doteq 2x \times AK$: ergo Radius Curvaturæ ad punctum *A* æqualis eſt *AK*; & eodem modo oſtenditur radium curvaturæ ad *Q* fore *QM*.

WPw, *ZAz* partes curvæ duæ exteriores ſecant omnes Hyperbolas, ad angulos rectos qui axem habent rectam *GD*. Partes duæ reliquæ interiores *WRw*, *ZQz* ad angulos rectos ſecant omnes Ellipſes qui axem habent *GD*. Adeoque ejuſdem curvæ portiones diverſæ ſatisfaciunt problemati in Ellipſi & Hyperbola; Ellipſis vero cujus axis minor coincidit cum axe Parabolæ, & centrum cum vertice, & cujus axis major eſt ad minorem ut Diameter Quadrati ad Latus, ſecabit omnes Parabolas circa axem illum eodem vertice deſcriptas.

FINIS.

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